RATIONALIZATION AND NONPARAMETRIC IDENTIFICATION OF DISCRETE GAMES WITH CORRELATED TYPES

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ABSTRACT. This paper studies the rationalization and identification of discrete games with players whose private information can be correlated. Our approach is fully nonparametric. First, under monotone pure strategy BNE, we characterize all the restrictions if any on the distribution of players' choices imposed by the game-theoretic model as well as restrictions associated with three assumptions that have been frequently used in the empirical analysis of discrete games. Namely, we consider additive separability of the private information in the payoffs, exogeneity of the payoff shifters relative to the private information, and mutual independence of the private information conditional on the payoff shifters. Second, we study the nonparametric identification of the payoff functions and type joint distribution under exclusion restrictions and rank conditions. In particular, we show that under the exogeneity assumption our structural model is identified up to a location-scale normalization in both nonseparable and separable cases. Last, we discuss possible estimation and testing procedures.

Keywords: Rationalization, Nonparametric Identification, Incomplete Information Games, Monotone Pure Strategy Bayesian Nash Equilibrium

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1. INTRODUCTION

Over the last decades, games with incomplete information have been much successful to understand the strategic interactions among agents in the analysis of various economic and social situations. A leading example is auctions with e.g. Vickrey (1961), Riley and Samuelson (1981), Milgrom and Weber (1982) for the theoretical side and Porter (1995), Guerre, Perrigne and Vuong (2000) and Athey and Haile (2002) for the empirical component. In this paper, we study the identification of static binary games of incomplete information where players have correlated types.[1] We also characterize all the restrictions if any imposed by such models on the observables, which are the players' choices. Following the work by Laffont and Vuong (1996) and Athey and Haile (2007) for auctions, our approach is fully nonparametric.

The empirical analysis of discrete games is almost thirty years old. In particular, the range of applications includes, among others, labor force participation (Bjorn and Vuong, 1984; Soetevent and Kooreman, 2007), firms' entry decisions ((Bresnahan and Reiss, 1990, 1991; Berry, 1992; Ciliberto and Tamer, 2009; Jia, 2008). These papers deal with discrete games under complete information. More recently, discrete games under incomplete information have been used to analyze social interactions by Brock and Durlauf (2001); Xu (2011), firm entry and location choices by Seim (2006), timing choices of radio stations commercials by Sweeting (2009), stock market analysts' recommendations by Bajari, Hong, Krainer, and Nekipelov (2010), capital investment strategies by Aradillas-Lopez (2010) and local grocery markets by Grieco (2011).

Our paper contributes to this literature in several aspects. First, we focus on monotone pure strategy Bayesian Nash equilibria (BNE) throughout. Monotonicity is a desirable property in many applications for both theoretical and empirical reasons. For instance, White, Xu, and Chalak (2011) show that monotone strategies are never worse off than non–monotone strategies in a private value auction model. Xu (2010) shows that the only equilibrium is a

[1] To simplify, we focus on binary games in this paper. We left the extension of our approach to general discrete games for future research.

monotone pure strategy BNE in an entry game when the strategic component coefficients are reasonably small. Moreover, the recent literature on nonparametric and nonseparable models heavily relies on the monotonicity relationship between observed variables and latent variables to establish identification results for structural functions (see, e.g., Chesher, 2003; Imbens and Newey, 2009; Matzkin, 2003). Chesher (2005) and Jun, Pinkse, and Xu (2010) exploit weak monotonicity in a triangular system with discrete endogenous variable to achieve partial and point identification for structural equations. On theoretical grounds, Athey (2001) provided the seminal result on the existence of a monotone pure strategy BNE whenever a Bayesian game obeys a Spence–Mirlees single–crossing restriction. Relying on the powerful notion of contractibility, Reny (2011) has extended Athey's results and related results by McAdams (2003) to give weaker conditions ensuring the existence of a monotone pure strategy BNE. Using Reny's results, we show that a monotone pure strategy BNE exists under a high-level assumption, namely, monotone expected payoffs in our setting. In particular, such a high-level assumption is satisfied if the game is of strategic complement and private information are positively regression dependent.

Second, we allow players' private information to be correlated. Allowing correlated private information is motivated primarily by empirical concerns. In particular, we do not require the mutual independence of private information across players, which is convenient but also imposes the strong restriction as we shall see that players' choices must be independent, which can be invalidated by the data.[2] Mutual independence, however, has been widely adopted in the empirical literature. See e.g. Brock and Durlauf (2001), Pesendorfer and Schmidt-Dengler (2003), Seim (2006), Aguirregabiria and Mira (2007), Sweeting (2009), Bajari, Hong, Krainer, and Nekipelov (2010), Tang (2010), Beresteanu, Molchanov, and Molinari (2011), Lewbel and Tang (2012) and De Paula and Tang (2012). Exceptions include Aradillas-Lopez (2010), Xu (2010), Wan and Xu (2010) and Liu and Xu (2012). When private information are correlated, the standard equilibrium concept – the pure strategy Bayesian

[2] A model featured with unobserved heterogeneity and independent private information also generates dependence among players' choices conditional on observed regressors (see Grieco, 2011).

Nash Equilibrium (BNE) – requires that each player's beliefs on rivals' choices depends on her private information. This means that each player's own private information is informative about rivals' equilibrium actions and each player makes adjustments to her beliefs on rivals' potential behaviors according to its realization. Alternatively, Aradillas-Lopez (2010) adopts another equilibrium concept, in which each player's equilibrium beliefs do not rely on her private information.

Third, our analysis is fully nonparametric in the sense that players' payoffs and the joint distribution of the players' private information are subject to some mild smoothness conditions only. As far as we know, with the exception of De Paula and Tang (2012) and Lewbel and Tang (2012), every paper analyzing empirical discrete games has imposed parametric restrictions on the payoffs and/or the distribution of the private information. For instance, Brock and Durlauf (2001), Seim (2006), Sweeting (2009) and Xu (2010) have specified both payoffs and the private information distribution parametrically. In a semi-parametric context, Aguirregabiria and Mira (2007) , Aradillas-Lopez (2010), Tang (2010), Wan and Xu (2010), Beresteanu, Molchanov, and Molinari (2011) and Liu and Xu (2012) parameterize players' payoffs, while Bajari, Hong, Krainer, and Nekipelov (2010) parameterize the private information distribution. On the other hand, De Paula and Tang (2012) and Lewbel and Tang (2012) do not introduce any parameter but impose some strong functional form restrictions on the payoffs. In particular, they impose multiplicative separability in the strategic effect and assume that it is a known function of the other players' choices. In addition to being fully nonparametric, we do not require either that players' private information enter additively in the payoffs. Consequently, our baseline discrete game is the most general one and closest to that considered in game theory. We show that such a model imposes essentially no restrictions on the distribution of players' choices. In other words, monotone pure strategy BNE can explain almost all observed choice probabilities in discrete games.

In view of the preceding result, we consider three assumptions that have been frequently used in the empirical analysis of discrete games. First, we consider the assumption that the private information enters additively in the player's payoff. To the best of our knowledge, such an assumption has been made in every paper analyzing discrete games empirically. We show again that the resulting model imposes essentially no restrictions on the distribution of players' choices. We also show that the players' payoffs and the joint distribution of the players' private information are not identified nonparametrically whether the private information are additively separable or not.

A second assumption that has been frequently imposed in empirical work is the exogeneity of some variables shifting the players' payoffs relative to players' private information. Papers using such an assumption are Brock and Durlauf (2001), Seim (2006), Sweeting (2009), Aradillas-Lopez (2010), Bajari, Hong, Krainer, and Nekipelov (2010), De Paula and Tang (2012), Lewbel and Tang (2012) and Liu and Xu (2012). We show that the resulting model restricts the distribution of players' choices conditional upon the payoff shifters and we characterize all those restrictions. Specifically, the exogeneity assumption restricts the joint choice probability to be a monotone function of the corresponding marginal choice probabilities. With the exogeneity assumption, we show that one can identify the belief of the player at the margin under a mild support condition. We then identify the players' payoffs and distribution of private information up to a scale-location normalization under some exclusion restrictions and rank conditions in both separable and nonseparable cases. Our identification result can be viewed as an extension to a game setup of the nonparametric identification results obtained by Matzkin (1992) for single–agent binary response models. An important difference is that the discrete game setup requires some exclusion restrictions for identification. Such restrictions have been used frequently in the empirical analysis of discrete games. See, e.g., Aradillas-Lopez (2010), Bajari, Hong, Krainer, and Nekipelov (2010), Tang (2010), Wan and Xu (2010), De Paula and Tang (2012) and Lewbel and Tang (2012).

For completeness, we consider a third assumption, namely the mutual independence of players' private information. Such an assumption has been used by several authors including Brock and Durlauf (2001), Seim (2006), Sweeting (2009), Bajari, Hong, Krainer, and Nekipelov (2010), Tang (2010), Beresteanu, Molchanov, and Molinari (2011), De Paula and Tang (2012) and Lewbel and Tang (2012). Specifically, we characterize all the restrictions imposed by exogeneity and mutual independence as considered by Brock and Durlauf (2001), Seim (2006), Sweeting (2009), Bajari, Hong, Krainer, and Nekipelov (2010), De Paula and Tang (2012) and Lewbel and Tang (2012). We show that the only restriction is that the players' choices are mutually independent conditionally on the payoff shifters. In particular, we show that the restrictions imposed by mutual independence are stronger than those imposed by exogeneity. In other words, exogeneity is redundant in terms of explaining players' choices as soon as mutual independence is imposed.

The paper is arranged as follows. We introduce our baseline model in Section 2. We define and establish the existence of a monotone pure strategy BNE. We also characterize such equilibrium strategies under additive separability of the private information. In Section 3, we study the restrictions imposed by the baseline model, whether the private information are additively separable or not. We also derive all the restrictions imposed by the exogeneity and mutual independence assumptions. In Section 4, we establish the identification of the belief of the player at the margin under exogeneity only. We then establish the nonparametric identification of the model primitives for the additively nonseparable and separable cases under some exclusion restrictions and rank conditions. Section 5 concludes with a discussion of the case when the same monotone pure strategy BNE is not played across identical games. We also discuss briefly estimation and testing in the nonseparable and separable cases.

2. Monotone Pure Strategy BNE

We consider a discrete game of incomplete information. There is a finite number of players, indexed by $i = 1, 2, \cdots, I$. Each player simultaneously chooses a binary action $Y_i \in \{0, 1\}$. Let $\mathcal{A} = \{0, 1\}^I$ be the space of possible actions for all players and $Y = (Y_1, \cdots, Y_I) \in \mathcal{A}$ be an action profile. Following the convention, let $\mathcal{A}_{-i}$ and Y_{-i} denote the action space and a profile of actions for all players except i, respectively. Let $X \in \mathscr{S}_X \subset \mathbb{R}^d$

be a payoff relevant random vector, which is publicly observed by all players and also by the researcher.[3] For instance, X can include individual characteristics of the players as well as specific variables for the game. For each player i, we further assume that the random variable $U_i \in \mathbb{R}$ is her private information which is not observed by other players. Let $U = (U_1, \cdots, U_I)$ and $F_{U|X}(\cdot|\cdot)$ be the conditional distribution function of U given the state variable X. The conditional distribution $F_{U|X}(\cdot|\cdot)$ is assumed to be common knowledge.[4]

The payoff for player i is described as

$$\Pi_i(Y, X, U_i) = \begin{cases} \pi_i(Y_{-i}, X, U_i), & \text{if } Y_i = 1, \\ 0, & \text{if } Y_i = 0, \end{cases}$$

where π_i is a structural function in our model. The zero payoff for action $Y_i = 0$ is a standard payoff normalization in binary response models.

Following the literature on Bayesian games, player i's decision rule is a function $Y_i = \delta_i(X, U_i)$, where $\delta_i(\cdot, \cdot) : \mathbb{R}^d \times \mathbb{R} \to \{0, 1\}$ maps all the information that she knows to the binary choice set. For $i = 1, \cdots, I$, let $a_{-i} = (a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_I) \in \mathcal{A}_{-i}$. Given a strategy profile $\delta = (\delta_1, \cdots, \delta_I)$, we denote by $\sigma^{\delta}_{-i}(a_{-i}|x, u_i)$ the conditional probability of others choosing $a_{-i} \in \mathcal{A}_{-i}$, i.e.,

$$\begin{aligned} \sigma^{\delta}_{-i}(a_{-i}|x, u_i) &\equiv \mathbb{P}_{\delta}\left(Y_{-i} = a_{-i}|X = x, U_i = u_i\right) \\ &= \int_{\mathbb{R}^{I-1}} \left[\prod_{j \neq i} \mathbf{1}\left\{\delta_j(x, u_j) = a_j\right\}\right] \times f_{U_{-i}|X,U_i}(u_{-i}|x, u_i) du_{-i} \end{aligned}$$

where $\mathbf{1}\{\cdot\}$ is the indicator function and $f_{U_{-i}|X,U_i}(\cdot|\cdot,\cdot)$ is the conditional density function of U_{-i} given X and U_i. Here $\mathbb{P}_{\delta}$ denotes the (conditional) probability measure under the strategy profile δ.

The equilibrium concept we adopt is the pure strategy Bayesian Nash equilibrium (BNE). Mixed strategy equilibria are not considered hereafter, since with probability one, each

[3] Grieco (2011) discusses the case with unobserved heterogeneity in publicly observed state variables.

[4] For a standard notion of common knowledge in game theory, see, e.g., Fudenberg and Tirole (1991), Chapter 14.

player has a unique best response in our case. Fix $X = x \in \mathscr{S}_X$. We now characterize BNEs in our discrete game. In equilibrium, player i with $U_i = u_i$ chooses action 1 if and only if her expected payoff is greater than zero, i.e.,

$$\delta_i^*(x, u_i) = \mathbf{1}\left[\sum_{a_{-i}} \pi_i(a_{-i}, x, u_i)\sigma_{-i}^*(a_{-i}|x, u_i) \geq 0\right], \tag{1}$$

where $\delta^* = (\delta_1^*, \cdots, \delta_I^*)$ is the equilibrium strategy profile and $\sigma_{-i}^*(a_{-i}|x, u_i)$ is a short notation for $\sigma_{-i}^{\delta^*}(a_{-i}|x, u_i)$. Note that σ_{-i}^* depends on δ_{-i}^*. Hence, eq. (1) for $i = 1, \cdots, I$ defines a simultaneous equation system in δ^* referred to as "mutual consistency". A pure strategy BNE is a fixed point δ^* of such a system, which holds for all $u = (u_1, \ldots, u_I)$ in the support $\mathscr{S}_{U|X=x}$. Ensuring equilibrium existence in Bayesian games is a complex and deep subject in the literature. It is well known that a solution of such an equilibrium generally exists in a broad class of Bayesian games (see, e.g., Vives, 1990).

Recently, much attention has focused on monotone pure strategy BNEs, since monotonicity is a desirable property in many applications such as auctions, entry, and global games. A monotone pure strategy BNE is defined as follows:

Definition 1 (Monotone pure strategy BNE)**.** *Fix $x \in \mathscr{S}_X$. A pure strategy profile $\delta^*(x) \equiv (\delta_1^*(x, \cdot), \cdots, \delta_I^*(x, \cdot))$, where $\delta_i^*(x, \cdot) : \mathscr{S}_{U_i|X=x} \to \{0, 1\}$, is a monotone pure strategy BNE if $\delta^*(x)$ is a BNE and $\delta_i^*(x, \cdot)$ is a monotone function on $\mathscr{S}_{U_i|X=x}$ for every $i = 1, \cdots, I$.*

Monotone pure strategy BNEs are easier to characterize than general BNEs. Fix $X = x$. In a monotone pure strategy BNE, players' strategies can be explicitly defined by a threshold profile $(u_1^*(x), \cdots, u_I^*(x))$ (recall that δ_i^* can take only two values, 0 or 1.) Formally, a monotone pure strategy BNE can be represented by a profile of cutoff values: $u^*(x) \equiv (u_1^*(x), \cdots, u_I^*(x)) \in \mathscr{S}_{U|X=x}$, where $u_i^*(\cdot) : \mathscr{S}_X \to \mathscr{S}_{U_i}$, such that $\delta_i^*(x, u_i) = \mathbf{1}[u_i \leq u_i^*(x)]$, or $\delta_i^*(x, u_i) = \mathbf{1}[u_i > u_i^*(x)]$.[5] Without loss of generality, we restrict our attention to monotone decreasing pure strategy BNEs hereafter. This serves as a normalization. To see this, suppose in a structure $[\pi; F_{U|X}]$, player i's equilibrium strategy is a monotone increasing

[5]The left–continuity of strategies considered hereafter is not restrictive given our assumptions below.

function for some fixed $x \in \mathscr{S}_X$, i.e. $\delta_i^*(x,\cdot) = \mathbf{1}[\cdot \geq u_i^*(x)]$. We can then construct an observationally equivalent structure $[\tilde{\pi}; \tilde{F}_{U|X}]$ by letting $\tilde{\pi}_i(\cdot, x, u_i) = \pi_i(\cdot, x, -u_i)$ for all $u_i \in \mathbb{R}$ with $\tilde{\pi}_j = \pi_j$ ($j \neq i$) and $\tilde{F}_{U|X}(\cdot|x) = F_{\widetilde{U}|X}(\cdot|x)$, where $\widetilde{U}$ differs from U only in its i–th argument: $\widetilde{U}_i = -U_i$. It can be shown that in equilibrium for the constructed structure, i's strategy is monotone decreasing, i.e. $\tilde{\delta}_i^*(x,\cdot) = \mathbf{1}[\cdot \leq -u_i^*(x)]$.

Given equilibrium monotone decreasing strategies of the form $\delta_i^*(x,u_i) = \mathbf{1}[u_i \leq u_i^*(x)]$ for $i = 1, \cdots, I$, the mutual consistency defined by eq. (1) for a BNE solution requires that $\forall x \in \mathscr{S}_X$,

$$u_i \leq u_i^*(x) \iff \mathbb{E}_{\delta^*}\left[\pi_i(Y_{-i}, X, U_i) \middle| X = x, U_i = u_i\right] \geq 0, \tag{2}$$

where $\mathbb{E}_{\delta^*}$ denotes the (conditional) expectation under the strategy profile δ^*. Without causing any confusion, we will suppress the subscript δ^* when it is an equilibrium strategy profile. In eq. (2), the conditional distribution of Y_{-i} given $X = x$ and $U_i = u_i$, i.e. $\mathbb{P}(Y_{-i} = a_{-i}|X = x, U_i = u_i)$, can be written as:

$$\sigma_{-i}^*(a_{-i}|x,u_i) = \mathbb{P}\left[\forall a_j = 1, U_j \leq u_j^*(X); \forall a_j = 0, U_j > u_j^*(X) \middle| X = x, U_i = u_i\right]. \tag{3}$$

Under Assumption R below, the $u_j^*(x)$s, if not on the support boundary, are defined by the set of simultaneous equations:

$$\sum_{a_{-i}} \pi_i(a_{-i}, x, u_i^*(x))\sigma_{-i}^*(a_{-i}|x, u_i^*(x)) = 0 \tag{4}$$

for $i = 1, \ldots, I$.

The seminal work on the existence of a monotone pure strategy BNE in games of incomplete information was first provided by Athey (2001) in both *supermodular* and *logsupermodular* games, and later extended by McAdams (2003) and Reny (2011). Applying Reny (2011) Theorem 4.1, we establish the existence of monotone pure strategy BNEs in our binary game under some weak regularity assumptions.

Assumption R (Conditional Radon–Nikodym Density)**.** *For every $x \in \mathscr{S}_X$, the conditional distribution of U given $X = x$ is absolutely continuous w.r.t. Lebesgue measure and has a continuous* *positive conditional Radon–Nikodym density $f_{U|X}(\cdot|x)$ a.e. over the nonempty interior of its hypercube support $\mathscr{S}_{U|X=x}$.*

Assumption R allows the support of U given $X = x$ to be bounded, namely of the form $\times_{i=1,\ldots,I}[\underline{u}_i(x), \overline{u}_i(x)]$ for some finite $\underline{u}_i(x)$ and $\overline{u}_i(x)$ as frequently used when U_i is i's private information, or unbounded as when $\mathscr{S}_{U|X=x} = \mathbb{R}^I$ used typically in binary models. Assumption R can be greatly weakened as shown by Reny (2011).

Assumption M (Monotone Decreasing Expected Payoff). *Fix an arbitrary $x \in \mathscr{S}_X$. For any monotone decreasing pure strategy profile δ, $\mathbb{E}_\delta\left[\pi_i(Y_{-i}, X, U_i)\middle|X = x, U_i = u_i\right]$ is a monotone decreasing function in $u_i \in \mathscr{S}_{U_i|X=x}$.*

Assumption M guarantees that the best response function is monotone decreasing in u_i if all other players adopt monotone decreasing pure strategies.

Lemma 1. *Suppose that Assumptions R and M hold. For any $x \in \mathscr{S}_X$, there exists a monotone decreasing pure strategy BNE.*

Proof. See Appendix A.1 □

Lemma 8 in Appendix A.2 provides some sufficient primitive conditions for Assumption M. Specifially, we assume positive regression dependence across U_is given X, strategic complementarity of players' actions and non-increasing payoffs in the U_is. Thus, monotone decreasing pure strategy BNEs generally exist in a large class of binary games. Note that Lemma 1 is silent about the existence of BNEs with non–monotone strategies. In a parametric setup of two–player binary games, Xu (2010) shows that non–monotone strategy BNEs can be ruled out under further restrictions on the strategic component coefficients and the correlation between private information. Lemma 1 does not ensure either that the monotone decreasing pure strategy BNE is unique. Throughout our analysis, we assume that only one monotone decreasing pure strategy BNE is played. In the Conclusion, we discuss the case when such an assumption is relaxed.

An assumption made in every paper in the empirical discrete game literature is the additive separability of the error terms in the payoffs.

Assumption S (Additive Separability). *We have* $\pi_i(a_{-i}, x, u_i) = \pi_i(a_{-i}, x) - u_i$ *for every* i, a_{-i}, x *and* u_i.

In Assumption S, the negative sign in front of u_i is only for notational convenience. Assumption S allows us to represent equilibrium strategies as a semi–linear–index binary response model as shown in the following lemma.

Lemma 2. *Suppose that Assumptions R, M and S hold. If a monotone decreasing pure strategy BNE is being played, i.e.,* $\delta^* = (\delta_1^*, \cdots, \delta_I^*)$ *where* δ_i^* *is a monotone decreasing function on* $\mathscr{S}_{U_i|X}$*, then equilibrium choices can be written as a semi–linear–index binary response model:*

$$Y_i = \mathbf{1}\left[U_i \leq \sum_{a_{-i}} \pi_i(a_{-i}, X)\sigma_{-i}^*(a_{-i}|X, u_i^*(X))\right], \tag{5}$$

Proof. See Appendix A.3. □

In particular, when $0 < \mathbb{E}(Y_i|X = x) < 1$ for all i, the profile of thresholds $u^*(x) = (u_1^*(x), \ldots, u_I^*(x))$ is a solution in $\times_{i=1}^I \mathscr{S}_{U_i|X=x}$ of the system of I equations:

$$\textstyle\sum_{a_{-i}} \pi_i(a_{-i}, x)\sigma_{-i}^*(a_{-i}|x, u_i^*(x)) = u_i^*(x),\ \forall i = 1, \cdots, I, \tag{6}$$

which is a special case of eq. (4).[6] The representation in eq. (5) of the equilibrium strategies as a semi–linear–index binary response model relates to single-agent binary threshold crossing models studied e.g. by Matzkin (1992). We will show in Section 4 that the belief of the player at the margin σ_{-i}^* in eq. (4) and (6) can be nonparametrically identified under additional weak conditions. The player at the margin is the one that receives a private information equal to the threshold $u_i = u_i^*(x)$ so that she is indifferent between action 1 and 0.

[6]In eq. (6) it is understood that $u_i^*(x) = \underline{u}_i(x)$ and $u_i^*(x) = \overline{u}_i(x)$ if $\mathbb{E}_{\delta^*}\left[\pi_i(Y_{-i}, X)\middle|X = x, U_i = \underline{u}_i(x)\right] < \underline{u}_i(x)$ and $\mathbb{E}_{\delta^*}\left[\pi_i(Y_{-i}, X)\middle|X = x, U_i = \overline{u}_i(x)\right] > \overline{u}_i(x)$, respectively.

3. RATIONALIZATION

In this section we will study the baseline model defined by Assumptions R and M as well as three other models obtained by imposing additional assumptions frequently made in the empirical game literature such as Assumption S. Specifically, we will characterize all the restrictions imposed on the distribution of observables (Y, X) by each of these models. We will say that a distribution of the observables is rationalized by a model if and only if it satisfies all the restrictions of the model. In other words, a distribution of the observables is rationalized if and only if there is a structure (not necessarily unique) in the model that generates such a distribution.

Besides Assumption S introduced above, we consider two additional assumptions. The first is the exogeneity of X relative to U, an assumption that has been frequently made in the empirical discrete game literature. See, e.g. Brock and Durlauf (2001), Seim (2006), Sweeting (2009), Aradillas-Lopez (2010), Bajari, Hong, Krainer, and Nekipelov (2010), De Paula and Tang (2012), Lewbel and Tang (2012) and Liu and Xu (2012).

Assumption E (Exogeneity)**.** *X and U are independent of each other.*[7]

Another assumption called as mutual independence has been widely used in the literature. See, e.g. Brock and Durlauf (2001), Seim (2006), Sweeting (2009), Bajari, Hong, Krainer, and Nekipelov (2010), Tang (2010), Beresteanu, Molchanov, and Molinari (2011), De Paula and Tang (2012) and Lewbel and Tang (2012).

Assumption I (Mutual Independence)**.** *$U_1, \cdots, U_I$ are mutually independent conditional on X.*

[7]Our results can be easily extended to the weaker assumption that Z and U are independent from each other conditional on W, where $X = (W, Z)$.

Let $S \equiv [\pi; F_{U|X}]$. We consider the following models (classes of structures):

$$\mathcal{M}_1 \equiv \big\{S : \text{Assumptions R and M hold and}$$
$$\text{a single monotone decreasing pure strategy BNE is played}\big\},$$
$$\mathcal{M}_2 \equiv \{S \in \mathcal{M}_1 : \text{ Assumption S holds}\},$$
$$\mathcal{M}_3 \equiv \{S \in \mathcal{M}_2 : \text{ Assumption E holds}\},$$
$$\mathcal{M}_4 \equiv \{S \in \mathcal{M}_3 : \text{ Assumption I holds}\}.$$

The last requirement in $\mathcal{M}_1$ is not restrictive when there is a unique monotone decreasing pure strategy BNE. When this is not the case, we follow most of the literature by assuming that the same equilibrium is played in the DGP for a given x. Relaxing such a requirement has been addressed in recent work and will be discussed in the Conclusion. Note that $\mathcal{M}_1 \supsetneq \mathcal{M}_2 \supsetneq \mathcal{M}_3 \supsetneq \mathcal{M}_4$.

We define some notation for our following discussion. Fix a structure $S \in \mathcal{M}_1$, let $\alpha_i(x) \equiv F_{U_i|X}(u_i^*(x)|x)$. Because equilibrium strategies are monotone decreasing, it is straightforward that $\alpha_i(x) = \mathbb{E}(Y_i|X = x)$. For every $p = 2, \cdots, I$, and $1 \leq i_1 < \cdots < i_p \leq I$, let $C_{U_{i_1},\cdots,U_{i_p}|X}(\cdot, \ldots, \cdot|\cdot)$ be the conditional copula function of $(U_{i_1}, \cdots, U_{i_p})$ given X, i.e. $\forall(\alpha_{i_1}, \cdots, \alpha_{i_p}) \in [0,1]^p$ and $x \in \mathscr{S}_X$,

$$C_{U_{i_1},\cdots,U_{i_p}|X}(\alpha_{i_1}, \cdots, \alpha_{i_p}|x) \equiv F_{U_{i_1},\cdots,U_{i_p}|X}\left(F^{-1}_{U_{i_1}|X}(\alpha_{i_1}|x), \cdots, F^{-1}_{U_{i_p}|X}(\alpha_{i_p}|x)\Big|x\right).$$

The next proposition determines distributions of Y given X that can be *rationalized* by a structure in $\mathcal{M}_1$.

Proposition 1. *A conditional distribution $F_{Y|X}(\cdot|\cdot)$ is rationalized by a structure in $\mathcal{M}_1$ if for every $x \in \mathscr{S}_X$ and $a \in \mathcal{A}$, $\mathbb{P}(Y = a|X = x) = 0$ implies that $\mathbb{P}(Y_i = a_i|X = x) = 0$ for some i.*

Proof. See Appendix B.1 □

Thus, $\mathcal{M}_1$ rationalizes all distributions for Y given X that belong to the interior of the simplex in $\mathbb{R}^{2^{I-k}}$ $(0 \leq k \leq I)$. In particular, $\mathcal{M}_1$ rationalizes all the distributions with strictly positive choice probabilities. Proposition 1 also indicates that the only possible distributions that cannot be rationalized by $\mathcal{M}_1$ satisfy $\mathbb{P}(Y = a|X = x) = 0$ for some $a \in \mathcal{A}$, i.e distributions for which there are "structural zeros." As a matter of fact, it is possible to characterize all those distributions that cannot be rationalized by $\mathcal{M}_1$.[8] They arise because of Assumption R. As noted earlier, one can replace the latter by Reny (2011)'s weaker assumptions. Lemma 9 in Appendix B.2 then shows that any distribution for Y given X can be rationalized. In other words, our binary game-theoretical model $\mathcal{M}_1$ imposes essentially no restrictions on the distribution of observables.

We now turn to the rationalization of $\mathcal{M}_2$. To do so, we establish the observational equivalence between $\mathcal{M}_1$ and $\mathcal{M}_2$ despite $\mathcal{M}_1 \supsetneq \mathcal{M}_2$.

Lemma 3. *For any given structure $S \equiv [\pi; F_{U|X}] \in \mathcal{M}_1$, there always exists an observationally equivalent structure $\tilde{S} \equiv [\tilde{\pi}; \tilde{F}_{U|X}] \in \mathcal{M}_2$.*

Proof. See Appendix B.3 □

The next proposition follows immediately from Lemma 3.

Proposition 2. *A conditional distribution $F_{Y|X}(\cdot|\cdot)$ is rationalized by a structure in $\mathcal{M}_1$ if and only if it is rationalized by a structure in $\mathcal{M}_2$.*

In particular, additive separability of private information in the payoffs (Assumption S) does not impose any additional restrictions relative to model $\mathcal{M}_1$. Moreover, it follows from Proposition 1 that $\mathcal{M}_2$ can still rationalize all distributions for Y given X that belong to the interior of the simplex in $\mathbb{R}^{2^{I-k}}$ with $0 \leq k \leq I$. In other words, $\mathcal{M}_2$ imposes essentially no restrictions on the distribution of observables.

Next, we consider the rationalization of $\mathcal{M}_3$. To do so, we first provide a necessary and sufficient condition for two structures in $\mathcal{M}_2$ to be observationally equivalent.

Lemma 4. *Two structures $S \equiv [\pi; F_{U|X}]$ and $\widetilde{S} \equiv [\tilde{\pi}; \tilde{F}_{U|X}]$ in $\mathcal{M}_2$ are observationally equivalent if and only if for every $x \in \mathscr{S}_X$,*

[8] Such a result is available upon request to the authors.

(i) $\forall\ i = 1, \cdots, I$, *we have* $\alpha_i(x) = \widetilde{\alpha}_i(x)$.

(ii) $\forall\ p = 2, \cdots, I$, *and* $1 \leq i_1 < \cdots < i_p \leq I$, *we have*

$$C_{U_{i_1},\cdots,U_{i_p}|X}\left(\alpha_{i_1}(x), \cdots, \alpha_{i_p}(x)|x)\big|x\right) = \widetilde{C}_{U_{i_1},\cdots,U_{i_p}|X}\left(\widetilde{\alpha}_{i_1}(x), \cdots, \widetilde{\alpha}_{i_p}(x)|x)\big|x\right).$$

Proof. See Appendix B.4. □

Lemma 4 characterizes the observational equivalence of two structures in $\mathcal{M}_2$ from two aspects: Condition (i) relates the conditional marginal distributions of U_i given X and the payoffs in the two structures; Condition (ii) equates the conditional dependence among the U_is given X between the two structures using the conditional copula function.

Based on Lemma 4, we can now give a necessary and sufficient condition for a structure in $\mathcal{M}_2$ to be observationally equivalent to a structure in $\mathcal{M}_3$.

Lemma 5. *For any given structure* $S \in \mathcal{M}_2$, *there exists an observationally equivalent structure* $\widetilde{S} \in \mathcal{M}_3$ *if and only if* $\forall p = 2, \cdots, I$, *and* $\forall\ 1 \leq i_1 < \cdots < i_p \leq I$

(i) $\forall x \in \mathscr{S}_X$, *we have* $C_{U_{i_1},\cdots,U_{i_p}|X}(\alpha_{i_1}(x), \cdots, \alpha_{i_p}(x)|x) = m_p(\alpha_{i_1}(x), \cdots, \alpha_{i_p}(x))$ *where*

$$\begin{aligned} & m_p(\alpha_{i_1}(x), \cdots, \alpha_{i_p}(x)) \\ & \quad \equiv \mathbb{E}\left[C_{U_{i_1},\cdots,U_{i_p}|X}\left(\alpha_{i_1}(X), \cdots \alpha_{i_p}(X)\big|X\right) |\alpha_{i_1}(X) = \alpha_{i_1}(x), \cdots, \alpha_{i_p}(X) = \alpha_{i_p}(x)\right] \quad (7) \end{aligned}$$

(ii) $m_p(\cdot, \cdots, \cdot)$ *is monotone strictly increasing on* $\mathscr{S}_{\alpha_{i_1}(X),\cdots,\alpha_{i_p}(X)}$ *except at values for which some coordinates are zero.*

(iii) $m_p(\cdot, \cdots, \cdot)$ *is continuously differentiable in the interior of* $\mathscr{S}_{\alpha_{i_1}(X),\cdots,\ \alpha_{i_p}(X)}$.

Proof. See Appendix B.5. □

Lemma 5 shows that condition (i) in Lemma 4 does not bind for a structure in $\mathcal{M}_2$ to be observationally equivalent to a structure in $\mathcal{M}_3$. It is due to the fact that the marginal choice probabilities generated by any structure in $\mathcal{M}_2$ can always be matched by I single-agent binary threshold crossing models. Specifically, for any $S \equiv (\pi; F_{U|X}(\cdot|x)) \in \mathcal{M}_2$ with cutoff values of $(u_1^*(x), \cdots, u_I^*(x))$, we can let $\tilde{\pi}_i(a_{-i}, x, u_i) = F_{U_i|X}(u_i^*(x)|x) - u_i$ and $\tilde{F}_{U_i|X}(\cdot|x)$

be the cdf of uniform distribution on $[0,1]$. It can be shown that the constructed structure $\tilde{S} \equiv (\tilde{\pi}; \tilde{F}_{U|X})$ generates the same marginal choice probabilities as the given structure S. Thus, Lemma 5 requires only conditions (i)–(iii) on the copula of the structure in $\mathcal{M}_2$. Such conditions arise as $m_p(\cdot, \cdots, \cdot)$ can be viewed as a copula in a model with exogenous payoff shifters, i.e. a model $\mathcal{M}_3$. For instance, conditions (ii)-(iii) follow from the properties of a copula.

We can now characterize all the restrictions imposed on the distribution of observables by model $\mathcal{M}_3$. Because model $\mathcal{M}_2$ does not impose any restriction by Proposition 1 and Proposition 2, these restrictions are due to Assumption E only. [9] Essentially, the following proposition translates the conditions (i)–(iii) in Lemma 5 in terms of observables.

Proposition 3. *A conditional distribution $F_{Y|X}(\cdot|\cdot)$ rationalized by a structure in $\mathcal{M}_2$ is also rationalized by a structure in $\mathcal{M}_3$ if and only if $\forall p = 2, \cdots, I$ and $\forall\ 1 \leq i_1 < \cdots < i_p \leq I$,*

(R1): $\forall\ x \in \mathscr{S}_X$, *we have*

$$\mathbb{E}\left(\prod_{j=1}^{p} Y_{i_j} \Big| X = x\right) = \mathbb{E}\left(\prod_{j=1}^{p} Y_{i_j} \Big| \alpha_{i_1}(X) = \alpha_{i_1}(x), \cdots, \alpha_{i_p}(X) = \alpha_{i_p}(x)\right).$$

(R2): $\mathbb{E}\left(\prod_{j=1}^{p} Y_{i_j} | \alpha_{i_1}(X) = \cdot, \cdots, \alpha_{i_p}(X) = \cdot\right)$ *is monotone strictly increasing on* $\mathscr{S}_{\alpha_{i_1}(X), \cdots, \alpha_{i_p}(X)}$ *except at values for which some coordinates are zero.*

(R3): $\mathbb{E}\left(\prod_{j=1}^{p} Y_{i_j} | \alpha_{i_1}(X) = \cdot, \cdots, \alpha_{i_p}(X) = \cdot\right)$ *is continuously differentiable in the interior of* $\mathscr{S}_{\alpha_{i_1}(X), \cdots, \alpha_{i_p}(X)}$.

Proof. See Appendix B.6. □

Proposition 3 shows that the joint choice probabilities rationalized by $\mathcal{M}_3$ are monotone strictly increasing and continuously differentiable functions of the marginal choice probabilities. Moreover, note that $\alpha_i(x) \equiv F_{U_i|X}(u_i^*(x)|x) = \mathbb{E}(Y_i|X = x)$ by monotone decreasing pure strategy BNE. Therefore, $\alpha_i(\cdot)$ is observable. It follows that the restrictions (R1)-(R3) are testable in principle. This is discussed further in the Conclusion.

[9] It also follows that the restrictions imposed by the set of structures with nonseparable private information and exogenous payoff shifters $\mathcal{M}_3' \equiv \{S \in \mathcal{M}_1 :\ \text{Assumption E holds}\}$ is given by Proposition 3.

For completeness, we study the restrictions on distribution of observables imposed by $\mathcal{M}_4 \equiv \{S \in \mathcal{M}_3 : \text{ Assumption I holds}\}$. Special cases of this model have been considered by several researchers using parametric or functional form restrictions. See, e.g. Brock and Durlauf (2001), Seim (2006), Sweeting (2009), Bajari, Hong, Krainer, and Nekipelov (2010), De Paula and Tang (2012) and Lewbel and Tang (2012). We first give a necessary and sufficient condition for a structure in $\mathcal{M}_3$ to be observationally equivalent to a structure in $\mathcal{M}_4$.

Lemma 6. *For an arbitrary given structure $S \in \mathcal{M}_3$, there exists an observationally equivalent structure $\widetilde{S} \in \mathcal{M}_4$ if and only if $\forall x \in \mathscr{S}_X$, $\forall p = 2, \cdots, I$, and $\forall\, 1 \leq i_1 < \cdots < i_p \leq I$, we have*

$$C_{U_{i_1},\cdots,U_{i_p}|X}\left(\alpha_{i_1}(x), \cdots, \alpha_{i_p}(x) \middle| x\right) = \prod_{j=1}^{p} \alpha_{i_j}(x). \tag{8}$$

Proof. See Appendix B.7. □

We note that as in Lemma 5, condition (8) involves only the copula but neither the marginal distributions nor the payoffs of the structure S. We also note that condition (8) is stronger than conditions (i)-(iii) of Lemma 5 together.

We can now characterize all the restrictions imposed on the distribution of observables by model $\mathcal{M}_4$. Because model $\mathcal{M}_2$ does not impose any restriction by Proposition 1 and Proposition 2, these restrictions are due to Assumptions E and I only. Essentially, the following proposition translates the restrictions in Lemma 5 and Lemma 6 in terms of observables. Since the restrictions in Lemma 5 are weaker than that in Lemma 6, as noted above, only the latter binds.

Proposition 4. *A conditional distribution $F_{Y|X}(\cdot|\cdot)$ rationalized by a structure in $\mathcal{M}_2$ is also rationalized by a structure in $\mathcal{M}_4$ if and only if $Y_1, \cdots, Y_I$ are conditionally independent given X.*

Proof. See Appendix B.8. □

As a matter of fact, a proof similar to that of Lemma 6 shows that condition (8) is also a necessary and sufficient condition for a structure in $\mathcal{M}_2$ to be observationally equivalent to a structure with separable and mutually independent private information, i.e. to a structure in $\mathcal{M}_4' \equiv \{S \in \mathcal{M}_2 : \text{ Assumption I holds}\}$. So it follows that Proposition 4 holds with $\mathcal{M}_4 \equiv \{S \in \mathcal{M}_2 : \text{ both Assumptions E and I hold}\}$ replaced by $\mathcal{M}_4'$. We summarize such discussion in the following corollary.

Corollary 1. *Model $\mathcal{M}_4$ imposes the same restrictions on distribution of players' choices as $\mathcal{M}_4'$, and both models are observationally equivalent.*

Because $\mathcal{M}_4 \subsetneq \mathcal{M}_4'$, exogeneity of the payoff shifters is redundant in terms of restrictions on the observables as soon as mutual independence of private information is imposed. [10]

4. Nonparametric Identification

In this section, we study the nonparametric identification of the baseline game-theoretical model $\mathcal{M}_1$, and its special cases $\mathcal{M}_2$, $\mathcal{M}_3$ and $\mathcal{M}_4$. As a preliminary to the identification of $\mathcal{M}_3$, we also consider the identification of $\mathcal{M}_3' \equiv \{S \in \mathcal{M}_1 : \text{ Assumption E holds}\}$ which is the nonseparable extension of $\mathcal{M}_3$.

4.1. Nonidentification of $\mathcal{M}_1$ and $\mathcal{M}_2$.

Proposition 5. *$\mathcal{M}_1$ and $\mathcal{M}_2$ are not identified nonparametrically.*

The nonidentification of $\mathcal{M}_1$ follows immediately from the observational equivalence between $\mathcal{M}_1$ and $\mathcal{M}_2$ established in Proposition 2. The nonidentification of $\mathcal{M}_2$ follows from the fact that there always exist I single-agent binary threshold crossing models matching the choice probabilities generated by any given structure in $\mathcal{M}_2$. Specifically, for any $S \equiv (\pi; F_{U|X}(\cdot|x)) \in \mathcal{M}_2$ with cutoff values of $(u_1^*(x), \cdots, u_I^*(x))$, we can let $\tilde{\pi}_i(a_{-i}, x, u_i) = u_i^*(x) - u_i$ and $\tilde{F}_{U|X}(\cdot|x) = F_{U|X}(\cdot|x)$. It is straightforward to see that the constructed structure $\tilde{S} \equiv (\tilde{\pi}; \tilde{F}_{U|X})$ generates the same choice probabilities as the given structure S. Thus, $\mathcal{M}_2$ is not identified.

[10] It also follows that the restrictions on distribution of observables imposed by the set of structures with nonseparable and mutually independent private information $\mathcal{M}_4'' \equiv \{S \in \mathcal{M}_1 : \text{ Assumption I holds}\}$ is given by Proposition 4.

We now turn to the identification of $\mathcal{M}_3'$ and $\mathcal{M}_3$. Our identification analysis proceeds in two steps: First, we establish the identification of the belief of the player at the margin, i.e. $\sigma^*_{-i}(a_{-i}|x, u_i^*(x))$, under weak conditions on the structures in $\mathcal{M}_1$. We then turn to the identification of the payoff π_i and the joint distribution $F_{U|X}$ of private information given X under additional conditions for the nonseparable case $\mathcal{M}_3'$ and the separable case $\mathcal{M}_3$, respectively. The most important conditions are the exclusion restrictions.

4.2. **Identification of $\sigma^*_{-i}(a_{-i}|x, u_i^*(x))$.** We make the following assumption where $\alpha(X) \equiv (\alpha_1(X), \cdots, \alpha_I(X))$.

Assumption RC–1 (Rank Condition)**.** *The support $\mathscr{S}_{\alpha(X)}$ of $\alpha(X)$ is a convex and compact subset of $[0,1]^I$ with $dim\big(\mathscr{S}_{\alpha(X)}\big) = I$.*

Because $\alpha_i(x) = \mathbb{E}(Y_i|X = x)$, condition RC–1 is verifiable. It requires that the payoff shifters X have a rich support whose dimension is equal to the number of players. This assumption is weak. For instance, when $I = 2$, this assumption is violated when $(\mathbb{E}(Y_1|X), \mathbb{E}(Y_2|X)$ lies on a line. Under Assumptions E and RC–1, the next proposition establishes the identification of $\sigma^*_{-i}(a_{-i}|x, u_i^*(x))$.

Lemma 7. *Fix $x \in \mathscr{S}_X$. Suppose that a structure $S \in \mathcal{M}_1$ satisfies Assumptions E and RC–1, then $\sigma^*_{-i}(\cdot|x, u_i^*(x))$ is identified.*

Proof. See Appendix B.9. □

Note that only the exogeneity assumption (with the rank condition) is used to identify the belief. In particular, Assumptions S and I are not required for the identification of the equilibrium belief σ^*_{-i}. This contrasts with the literature, see e.g. Lewbel and Tang (2012). For future use, we define the vector $\Sigma^*_{-i}(x) \in \mathbb{R}^{2^{I-1}}$ whose j–th argument equals to $\sigma^*_{-i}\left(a^{(j)}_{-i}|x, u_i^*(x)\right)$ where $a^{(j)}_{-i}$ is the j–th element in $\mathcal{A}_{-i}$.

4.3. **Identification of $\mathcal{M}_3'$.** First, it should be noted that without further restrictions on the structure, the payoffs are not identified by the proof of Lemma 5 for a structure in $\mathcal{M}_3$, i.e.

we can show that any $S \in \mathcal{M}_3$ is observationally equivalent to another structure $\widetilde{S} \in \mathcal{M}_3$ corresponding to I single-agent binary threshold models. Thus, to identify the payoffs, we impose some exclusion restrictions as in Aradillas-Lopez (2010), Bajari, Hong, Krainer, and Nekipelov (2010), Tang (2010), Wan and Xu (2010), De Paula and Tang (2012) and Lewbel and Tang (2012).

Assumption ER (Exclusion Restriction)**.** *Let $X = (X_1, \cdots, X_I)$. For all i, a_{-i}, x and u_i, we have $\pi_i(a_{-i}, x, u_i) = \pi_i(a_{-i}, x_i, u_i)$.* [11]

Moreover, we make the following assumptions.

Assumption C. *Fix $x \in \mathscr{S}_X$. The function $\pi_i(a_{-i}, x_i, u_i)$ is continuous in u_i for every i and a_{-i}.*

Assumption RC–2 (Rank Condition)**.** *Fix $x \in \mathscr{S}_X$. The matrix $\mathbb{E}\left[\Sigma^*_{-i}(X)\Sigma^*_{-i}(X)^\top \middle| X_i = x_i, \alpha_i(X) = \alpha_i\right]$ has a rank that equals to $2^{I-1} - 1$, for every i and $\alpha_i \in \mathscr{S}_{\alpha_i(X)|X_i=x_i} \cap (0,1)$.*

When π_i is additively separable in u_i, Assumption C is trivially satisfied. Assumption RC–2 requires that conditional on X_i and $\alpha_i(X)$, X_{-i} varies sufficiently to cause enough variation in $\sigma^*_{-i}(\cdot|x, u^*_i(x))$. Such assumption is satisfied in the two-player case when players' choices are not degenerate.

Note that if the matrix $\mathbb{E}\left[\Sigma^*_{-i}(X)\Sigma^*_{-i}(X)^\top \middle| X_i = x_i, \alpha_i(X) = \alpha_i\right]$ is full rank equal to 2^{I-1} for some $\alpha_i \in (0,1)$, then by the proofs of Proposition 6 one can verify that the structural function $\pi_i\big(a_{-i}, x_i, q_i(\alpha_i)\big) = 0$ for all $a_{-i} \in \mathcal{A}_{-i}$ and then, under Assumption M, $\mathscr{S}_{\alpha_i(X)|X_i=x_i} = \{\alpha_i\}$. This degenerated case corresponds to the special scenario that in equilibrium there is no strategic effects from player i's rivals when $X_i = x_i$. Therefore, if $\pi_i\big(a_{-i}, x_i, q_i(\alpha_i)\big) \neq \pi_i\big(a'_{-i}, x_i, q_i(\alpha_i)\big)$ for some $a_{-i}, a_{-i} \in \mathcal{A}_{-i}$, then $2^{I-1} - 1$ is the largest possible rank that the matrix $\mathbb{E}\left[\Sigma^*_{-i}(X)\Sigma^*_{-i}(X)^\top \middle| X_i = x_i, \alpha_i(X) = \alpha_i\right]$ could have.

Let $q_i(\cdot) \equiv F^{-1}_{U_i}(\cdot)$ be the quantile function of the marginal distribution of U_i.

Proposition 6. *Fix $x_i \in \mathscr{S}_{X_i}$. Let $S \in \mathcal{M}'_3$. Suppose that assumptions RC–1, ER, C and RC–2 hold, then for every $\alpha_i \in \mathscr{S}_{\alpha_i(X)|X_i=x_i} \cap (0,1)$, $\pi_i\big(\cdot, x_i, q_i(\alpha_i)\big)$ is identified up to to the scale. In*

[11] As a matter of fact, X_is can have some common variables. To simplify, we assume that X_is partition X.

addition, if there exists $x' = (x_i, x'_{-i}) \in \mathscr{S}_X$ such that $\alpha_i(x') \neq \alpha_i$ and $(\alpha_i, \alpha_{-i}(x')) \in \mathscr{S}_{\alpha(X)}$, then the sign of $\pi_i(\cdot, x_i, q_i(\alpha_i))$ is also identified.

Proof. See Appendix B.11. □

Proposition 6 shows that the players' nonseparable payoffs are identified up to scale as well as up to the marginal distributions of players' private information, on an appropriate domain which is essentially the support of the F_{U_i}-quantile associated with $\mathbb{E}(Y_i|X)$ controlling for $X_i = x_i$. The more variations in $\mathbb{E}(Y_i|X)$ when X_{-i} varies, the larger will be this domain. This domain excludes the boundaries $q_i(0)$ and $q_i(1)$ for technical reasons. For the purpose of generality, in Proposition 6, we allow a "zero" scale, as well as positive and negative values. "Zero" scale could occur only if the support $\mathscr{S}_{\alpha_i(X)|X_i=x_i}$ is a singleton.

Now we turn to the identification of F_U. Knowledge of F_U is equivalent to the knowledge of its copula C_U and its marginals F_{U_i}, $i = 1, \cdots, I$. The copula C_U is identified on the support $\mathscr{S}_{\alpha(X)}$ of $\alpha(X)$ due to

$$C_U(\alpha_1(x), \cdots, \alpha_I(x)) = F_U(u_1^*(x), \cdots, u_I^*(x)) = \mathbb{P}(Y_1 = \cdots = Y_I = 1|X = x) \qquad (9)$$

for every $x \in \mathscr{S}_X$. On the other hand, the marginal distributions are not identified. This is expected in view of Matzkin (2003) results for nonseparable models.

The discussion above shows that, for normalization purposes, we are free to choose the scale of the payoffs and the marginal distributions of private information F_{U_i}. One straightforward normalization is given by normalizing the scale of the payoffs and the marginal distributions of private information F_{U_i}. Under such a normalization, $\pi_i(a_{-i}, x_i, \cdot)$ is then identified on the domain $q_i\left(\mathscr{S}_{\alpha_i(X)|X_i=x_i} \cap (0,1)\right)$ for every $a_{-i} \in \mathcal{A}_{-i}$ by Proposition 6.

4.4. **Identification of $\mathcal{M}_3$.** Next, we consider the case where additive separability of private information in payoffs is imposed in $\mathcal{M}_3'$, i.e. model $\mathcal{M}_3$. With additive separability of private information in the payoffs, we obtain a stronger identification result under weaker rank condition in $\mathcal{M}_3$ than $\mathcal{M}_3'$. We give such a rank condition as follows. Let $\overline{\Sigma}^*_{-i}(X) \equiv \Sigma^*_{-i}(X) - \mathbb{E}\left[\Sigma^*_{-i}(X)|X_i, \alpha_i(X)\right]$.

Assumption RC–3 (Rank Condition)**.** *The matrix* $\mathbb{E}\left[\overline{\Sigma}^*_{-i}(X)\overline{\Sigma}^*_{-i}(X)^\top \middle| X_i = x_i\right]$ *has a rank of* $2^{I-1}-2$ *for all i and a fixed* $x \in \mathscr{S}_X$ *such that* $\Big(\alpha_1(x), \cdots, \alpha_I(x)\Big) \in (0,1)^I$.

Assumption RC–3 requires that conditional on X_i, X_{-i} varies sufficiently to cause enough variation in the de-meaned belief of the player at the margin $\overline{\sigma}^*_{-i}(\cdot|x, u^*_i(x)) \equiv \sigma^*_{-i}(\cdot|x, u^*_i(x)) - \mathbb{E}\left[\sigma^*_{-i}(\cdot|X, u^*_i(X))|X_i = x_i, \alpha_i(X) = \alpha_i(x)\right]$. Similarly, a full rank ($2^{I-1}-1$) of the matrix $\mathbb{E}\left[\overline{\Sigma}^*_{-i}(X)\overline{\Sigma}^*_{-i}(X)^\top \middle| X_i = x_i\right]$ only occurs when there is no strategic effects on i with $X_i = x_i$. Assumption RC–3 is trivially satisfied in the two-player case when players' choices are not degenerate. Such assumption is, in general, weaker than Assumption RC–2 because Assumption RC–2 requires that the rank condition holds for all $\alpha_i \in \mathscr{S}_{\alpha_i(X)|X_i=x_i}$.

The following proposition gives the identification result of players' payoffs in model $\mathcal{M}_3$.

Proposition 7. *Fix* $x_i \in \mathscr{S}_{X_i}$ *such that* $\alpha_i(x) \in (0,1)$ *for some* $x \in \mathscr{S}_{X|X_i=x_i}$. *Let* $S \in \mathcal{M}_3$. *Suppose that Assumptions RC–1, ER, and RC–3 hold, then* $h_i(\cdot, x_i) \equiv \pi_i(\cdot, x_i) - \pi_i(a^0_{-i}, x_i)$ *is identified up to scale. In addition, if there exists* $x, x' \in \mathscr{S}_{X|X_i=x_i}$ *such that* $\alpha_i(x) \neq \alpha_i(x')$ *and* $(\alpha_i(x), \alpha_{-i}(x')) \in \mathscr{S}_{\alpha(X)}$, *then the sign of* $h_i(\cdot, x_i)$ *is also identified.*

Proof. See Appendix B.12. □

Proposition 7 shows that the players' additively separable payoffs are identified up to a location and a scale on the whole support of the private information.[12] Such identification result is stronger than the one in Proposition 6 due to the additive separability of private information in payoffs.

We then turn to the identification of the joint distribution of private information F_U. Identification of F_U is equivalent to identify both its copula C_U and its marginals F_{U_i}, $i = 1, \cdots, I$. As shown in Section 4.3, C_U is identified on the support $\mathscr{S}_{\alpha(X)}$ of $\alpha(X)$ by eq. (9). In addition, the marginals F_{U_i}, $i = 1, \cdots, I$, are identified up to a location and a scale.

[12]Based on Proposition 7, Lemma 10 in Appendix B.14 provides a necessary and sufficient condition for two structures in $\mathcal{M}_3$ to be observationally equivalent.

To see this, we first look at the following equilibrium condition

$$u_i^*(x) = \sum_{a_{-i} \in \mathcal{A}_{-i}} \pi_i(a_{-i}, x_i) \times \sigma_{-i}^*(a_{-i}|x, u_i^*(x)).$$

It implies that $u_i^*(x)$ is identified up to a location and scale by Proposition 7 since the belief of the player at the margin $\sigma_{-i}^*(\cdot|x, u_i^*(x))$ is identified by Lemma 7. Consequently, the marginal quantile function $F_{U_i}^{-1}$ is identified up to a location and scale on the support $\mathscr{S}_{\alpha_i(X)}$ of $\alpha_i(X)$ due to $F_{U_i}^{-1}(\mathbb{E}(Y_i|X=x)) = u_i^*(x)$.

The discussion above shows that, for normalization purposes, we are free to choose a location and a scale for each player. One convenient normalization is to normalize the scale of payoffs and one quantile of the marginal distribution F_{U_i} of private information. Such a normalization pins down both the location and scale. By Proposition 7, the players' payoffs are then identified under this normalization. Our discussion in the above paragraph also shows that the marginal distributions of private information are identified on an appropriate domain in this case.

Corollary 2. *Fix $x_i \in \mathscr{S}_{X_i}$ such that $\mathscr{S}_{\alpha_i(X)|X_i=x_i}$ is not a singleton. Let $S \in \mathcal{M}_3$. Suppose $h_i(\cdot, x_i)$ is identified up to scale and the sign of $h_i(\cdot, x_i)$ is also identified under the conditions in Proposition 7. Let further median$(U) = 0 \in \mathscr{S}_{\alpha_i(X)|X=x_i}$ and $\|\pi_i(\cdot, x_i)\| = 1$ for all $x_i \in \mathscr{S}_{X_i}$. Then $\pi_i(\cdot, x_i)$ is identified.*

Proof. See Appendix B.13. □

Remark 1. *We can have an alternative strategy to identify players' payoffs in model $\mathcal{M}_3$. It is based on the following single–index structure,*

$$\begin{aligned}\mathbb{E}(Y_i|X_i = x_i, \Sigma_{-i}^*(X) = \Sigma_{-i}^*(x)) \\ = F_{U_i}\left(\pi_i\left(a_{-i}^0, x_i\right) + \sum_{a_{-i} \in \mathcal{A}/\{a_{-i}^0\}} h_i(a_{-i}, x_i)\sigma_{-i}^*\left(a_{-i}|x, u_i^*(x)\right)\right)\end{aligned} \quad (10)$$

where the belief vector of the player at the margin $\Sigma^*_{-i}(x)$ is identified for any $x \in \mathscr{S}_X$ due to Lemma 7. We can identify $h_i(a_{-i}, x_i) \equiv \pi_i(a_{-i}, x_i) - \pi_i(a^0_{-i}, x_i)$ up to scale by differentiating eq. (10) with respect to $\sigma^*_{-i}(a_{-i}|x, u^*_i(x))$ for any $a_{-i} \in \mathcal{A}_{-i}$. Thus, players' payoffs $\pi_i(\cdot, x_i)$ are identified up to location and scale. This strategy, however, involves a support condition (on $\Sigma^*_{-i}(X)$ given $X_i = x_i$), which is stronger than the one used in the strategy above.

5. CONCLUSION

This paper addresses the rationalization and identification of discrete games with correlated private private information in a fully nonparametric way. We show that our baseline game-theoretical model does not impose any restriction on observables. This implies that binary bayesian games are not testable in view of players' choices only. We also characterize all the restrictions on players' choices imposed by three assumptions frequently made in the empirical analysis of discrete games. We then exploit exclusion restrictions to identify our structural model nonparametrically in both nonseparable and separable cases. These restrictions take the form of excluding part of a player's payoff shifters from all other players' payoffs as frequently assumed in the empirical discrete game literature.

We require that the same monotone pure strategy BNE is played in the DGP for a given x. The nonparametric analysis relaxing such a requirement clearly needs to be developed. In particular, all of our rationalization and identification results will be weakened in that situation. In particular, $\mathcal{M}_1$ and $\mathcal{M}_2$ still impose no restrictions on observables whether or not a single monotone pure strategy BNE is played. On the other hand, $\mathcal{M}_3$ and $\mathcal{M}_4$ will impose weaker restrictions than those in Proposition 3 and Proposition 4, respectively. Location/scale identification of these two models, as established in Proposition 7, will be lost. Thus, point identification of the primitives would require additional identifying assumptions. Alternatively, one could follow the set-identification approach initiated by Tamer (2003) in the presence of multiple BNE.

A second line of research, which needs to be developed, concerns model testing. Our Proposition 3 and Proposition 4 become especially useful as they characterize all the restrictions in terms of observables imposed by $\mathcal{M}_3$ and $\mathcal{M}_4$. Thus such restrictions are in principle testable. In particular, some tests can be relatively easy to develop as they only involve some nonparametric regressions. For instance, the restriction given in Proposition 4 can be tested by using conditional independence tests developed in statistics and econometrics (see, e.g. Su and White (2007) and Su and White (2008)). It is also worthnoting that such tests do not rely on identification of the model and consequently on the assumptions used to identify the primitives.

Lastly, a third line of research deals with the nonparametric estimation of the various models. In a semiparametric setup, Liu and Xu (2012) propose an estimation procedure for our model $\mathcal{M}_3$ with linear payoff, and establish the root-N consistency of the linear payoff coefficients. A fully nonparametric estimation, however, deserves future investigation. A strategy could rely on the identification results and propose a sample-analog type of estimators for the players' payoffs and the joint distribution of private information. Establishing the asymptotic properties of such an estimation procedure is left for future research. The main difficulty relies on the generated covariates, namely the belief of the player at the margin which appears in the expected payoff. Such a problem could be addressed by using the most recent literature on nonparametric regression with nonparametrically generated covariates (see, e.g., Mammen, Rothe, and Schienle (2012)).

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Appendix A. Proofs of Results in Section 2

A.1. **Proof of Lemma 1.** First, Assumptions G1–G6 of Reny (2011) are satisfied in our discrete game under Assumption R. Moreover, by Assumption M, when other players employ monotone decreasing pure strategies, player i's best response is also a monotone decreasing pure strategy, which is unique. By Reny (2011, Theorem 4.1), the conclusion follows. $\square$

A.2. **Existence of monotone pure strategy BNEs under primitive conditions.**

Definition 2. *a set $A \subseteq \mathbb{R}^d$ is upper if and only if its indicator function is non–decreasing, i.e., for any $x, y \in \mathbb{R}^d$, $x \in A$ and $x \leq y$ imply $y \in A$, where $x \leq y$ means $x_i \leq y_i$ for $i = 1, \cdots, d$.*

Assumption A (Positive regression dependence)**.** *For any $x \in \mathscr{S}_X$ and any upper set $A \subseteq \mathbb{R}^{I-1}$, the conditional probability $\mathbb{P}\left(U_{-i} \in A | X = x, U_i = u_i\right)$ is non–decreasing in $u_i \in \mathscr{S}_{U_i|X=x}$.*

Assumption B (Strategic complement)**.** *For any $x \in \mathscr{S}_X$ and $u_i \in \mathscr{S}_{U_i|X=x}$, suppose $a_{-i} \leq a'_{-i}$, then $\pi_i(a_{-i}, x, u_i) \leq \pi_i(a'_{-i}, x, u_i)$.*

Assumption C (Non-increasing Payoffs)**.** *$\forall\ i = 1, \cdots, I$ and $\forall\ (a_{-i}, x) \in \mathcal{A}_{-i} \times \mathscr{S}_X$, $\pi_i(a_{-i}, x, \cdot)$ are non-increasing functions in $u_i \in \mathscr{S}_{U_i|X=x}$.*

Lemma 8. *Suppose that assumptions R, A, B and C hold. For any $x \in \mathscr{S}_X$, there exists a monotone decreasing pure strategy BNE.*

Proof. By Lemma 1, it suffices to show that Assumption M holds. Fix $x \in \mathscr{S}_X$. Given an arbitrary monotone decreasing pure strategy profile: $\delta_i(x, u_i) = \mathbf{1}[u_i \leq u_i(x)]$ for $i = 1, \cdots, I$, where $u_i(x) \in \mathscr{S}_{U_i|X=x}$. By assumptions A and B, and Lehmann (1955), for any $u_i < u'_i$ in $\mathscr{S}_{U_i|X=x}$, we have

$$\mathbb{E}_\delta\left[\pi_i(Y_{-i}, X, u_i) | X = x, U_i = u'_i\right] \leq \mathbb{E}_\delta\left[\pi_i(Y_{-i}, X, u_i) | X = x, U_i = u_i\right].$$

Further, by assumption C,

$$\mathbb{E}\delta\left[\pi_i(Y_{-i}, X, u'_i) | X = x, U_i = u'_i\right] \leq \mathbb{E}_\delta\left[\pi_i(Y_{-i}, X, u_i) | X = x, U_i = u'_i\right].$$

Thus, $\mathbb{E}_\delta\left[\pi_i(Y_{-i}, X, U_i) | X = x, U_i = u_i\right]$ is a non–increasing function of u_i. $\square$

A.3. **Proof of Lemma 2.** Fix $X = x$. By Assumption S, there is $\mathbb{E}_{\delta^*}\left[\pi_i(Y_{-i}, X, U_i)\middle|X = x, U_i = u_i\right] = \mathbb{E}_{\delta^*}\left[\pi_i(Y_{-i}, X)\middle|X = x, U_i = u_i\right] - u_i$. Because $\mathbb{E}_{\delta^*}\left[\pi_i(Y_{-i}, X)\middle|X = x, U_i = u_i\right]$ is a linear combination of $\sigma^*_{-i}(a_{-i}|x, u_i)$ for all $a_{-i} \in \mathcal{A}_{-i}$, which are continuous in u_i under Assumption R, then $\mathbb{E}_{\delta^*}\left[\pi_i(Y_{-i}, X, U_i)\middle|X = x, U_i = u_i\right]$ is a continuous and monotone decreasing function in u_i under Assumption M. Hence, the cutoff value $u_i^*(x)$ defining player i's equilibrium strategy satisfies: if $\underline{u}_i(x) < u_i^*(x) < \overline{u}_i(x)$, we have

$$\mathbb{E}_{\delta^*}\left[\pi_i(Y_{-i}, X)\middle|X = x, U_i = u_i^*(x)\right] - u_i^*(x) = 0,$$

which implies that: conditional on $\underline{u}_i(X) < u_i^*(X) < \overline{u}_i(X)$, there is

$$Y_i = \mathbf{1}\left[U_i \leq u_i^*(X)\right] = \mathbf{1}\left[U_i \leq \sum_{a_{-i}} \pi_i(a_{-i}, X)\sigma^*_{-i}\big(a_{-i}|X, u_i^*(X)\big)\right].$$

If $u_i^*(x) = \overline{u}_i(x)$, then $\mathbb{E}_{\delta^*}\left[\pi_i(Y_{-i}, X)\middle|X = x, U_i = \overline{u}_i(x)\right] - \overline{u}_i(x) \geq 0$, which implies that: conditional on $u_i^*(x) = \overline{u}_i(x)$, there is

$$Y_i = \mathbf{1}\left[U_i \leq \overline{u}_i(X)\right] \leq \mathbf{1}\left[U_i \leq \sum_{a_{-i}} \pi_i(a_{-i}, X)\sigma^*_{-i}\big(a_{-i}|X, \overline{u}_i(X)\big)\right].$$

Because $\mathbf{1}\left[U_i \leq \overline{u}_i(X)\right] = 1$ a.s., thus

$$Y_i = \mathbf{1}\left[U_i \leq \overline{u}_i(X)\right] = \mathbf{1}\left[U_i \leq \sum_{a_{-i}} \pi_i(a_{-i}, X)\sigma^*_{-i}\big(a_{-i}|X, \overline{u}_i(X)\big)\right] \quad a.s..$$

Similar arguments hold for the case $u_i^*(x) = \underline{u}_i(x)$. □

Appendix B. Proofs of Results in Section 3

B.1. **Proof of Proposition 1.** Fix $x \in \mathscr{S}_X$. First, we assume $\mathbb{P}(Y = a|X = x) > 0$ for all $a \in \mathcal{A}$. Now we construct a structure in $\mathcal{M}_1$ to rationalize $F_{Y|X}(\cdot|x)$. Let $\pi_i(a_{-i}, x, u_i) = \mathbb{E}(Y_i|X = x) - u_i$ for $i = 1, \cdots, I$. Note that there is no strategic effect by construction and Assumption M is satisfied. Now we construct $F_{U|X}(\cdot|x)$. Let $F_{U_i|X}(\cdot|x)$ be uniformly distributed on $[0, 1]$. So it suffices to construct the copula function $C_{U|X}(\cdot|x)$. Let $C_{U|X}(\alpha_1, \cdots, \alpha_I|x) = 0$ if $\alpha_i = 0$ for some i. Then only restriction left for constructing such a copula is: on the support $\{\mathbb{E}(Y_1|X = x), 1\} \times \cdots \times \{\mathbb{E}(Y_I|X = x), 1\}$, let $C_{U|X}(\alpha_1, \cdots, \alpha_I|x) = \mathbb{E}(\prod_{j=1}^{p} Y_{i_j}|X = x)$ where $i_1, \cdots, i_p$ are all the indexes such that $\alpha_{i_j} = \mathbb{E}(Y_{i_j}|X = x)$. On this sub–support, it is straightforward that $C_{U|X}(\cdot|x)$ is monotone increasing in each index by the fact that $\mathbb{P}(Y = a|X = x) > 0$ for all $a \in \mathcal{A}$. Thus it is straightforward that we can extend $C_{U|X}(\cdot|x)$ to the whole support $[0,1]^I$ such that $C_{U|X}(\cdot|x)$ is monotone increasing and smooth. It can be shown that the given conditional distribution of Y given $X = x$ can be rationalized by this constructed structure in $\mathcal{M}_1$: $\mathbb{P}(Y_{i_1} = 1, \cdots, Y_{i_p} = 1|X = x) = C_{U_{i_1},\cdots,U_{i_p}|X}(\mathbb{E}(Y_{i_1}|X = x), \cdots, \mathbb{E}(Y_{i_p}|X = x))$ for any subset index $\{i_1, \cdots, i_p\}$.

When $\mathbb{P}(Y = a|X = x) = 0$ for some a's in $\mathcal{A}$. By the condition in Proposition 1, the conditional distribution of Y given $X = x$ is degenerated in some indexes. W.l.o.g., let $\{1, \cdots, k\}$ be set of indexes such that $\mathbb{P}(Y_i = 1|X = x) = 0$ or 1; and $\{k+1, \cdots, I\}$ satisfying $0 < \mathbb{P}(Y_i = 1|X = x) < 1$. Then let $\pi_i(a_{-i}, x, u_i) = \mathbb{E}(Y_i|X = x) - u_i$ for $i = 1, \cdots, I$. For player $i = k+1, \cdots, I$, we can construct a copula function $C_{U_{k+1},\cdots,U_I|X}(\cdot|x)$ as described above such that $C_{U_{k+1},\cdots,U_I|X}(\cdot|x)$ is monotone increasing and smooth. Similarly, the constructed structure rationalizes the given conditional distribution of Y given $X = x$. □

B.2. **Rationalizing All Probability Distributions.** Suppose we replace Assumption R with the following conditions in Reny (2011): For every $x \in \mathscr{S}_X$,

G.2. The distribution $F_{U_i|X}(\cdot|x)$ on $\mathscr{S}_{U_i|X=x}$ is atomless.

G.3. There is a countable subset $\mathscr{S}^0_{U_i|X=x}$ of $\mathscr{S}_{U_i|X=x}$ such that every set in $\mathscr{S}_{U_i|X=x}$ assigned positive probability by $F_{U_i|X}(\cdot|x)$ contains two points between which lies a point in $\mathscr{S}^0_{U_i|X=x}$.

Note that it is straightforward that Assumptions G.1 and G.4 through G.6 in Reny (2011) are all satisfied in our discrete game because the action space $\mathcal{A}$ is finite and the conditional distribution of U given $X = x$ has a hypercube support in $\mathbb{R}^I$. Thus, the conclusion in Lemma 1 still holds (i.e., existence of a monotone pure strategy BNE) under Assumptions G.2, G.3 and M. Moreover, if we define

$$\mathcal{M}'_1 \equiv \{S : \text{G.2, G.3 and M hold and a single monotone decreasing pure strategy BNE is played}\}$$

Then we can show the following result:

Lemma 9. *For any $x \in \mathscr{S}_X$, $F_{Y|X}(\cdot|x)$ can be rationalized by a structure $S \equiv [\pi_1, \cdots, \pi_I; F_{U|X}] \in \mathcal{M}'_1$.*

Proof. Fix $x \in \mathscr{S}_X$. Now we construct a structure in $\mathcal{M}'_1$ to rationalize $F_{Y|X}(\cdot|x)$. Let $\pi_i(a_{-i}, x, u_i) = \alpha_i(x) - u_i$ for $i = 1, \cdots, I$. Note that there is no strategic effect by construction and Assumption M is satisfied. Now we construct $F_{U|X}(\cdot|x)$. Let $[0,1]^I$ be the support of the distribution and partition it into 2^I disjoint events: $\bigotimes_{i=1}^{I}\{[0,\alpha_i(x)),[\alpha_i(x),1]\}$ [13]. For each event B_j, $(j=1,\cdots,2^I)$, we define a conditional distribution $F_{U|X=x,U\in B_j}$ as a uniform distribution on B_j, where B_j is the j–th event in the partition of the support. Moreover, let $\mathbb{P}(U\in B_j|X=x)=\mathbb{P}(Y=a(j)|X=x)$ where $a(j)\in\mathcal{A}$ and satisfies $a_i(j)=0$ if the i–th argument of event B_j is $[\alpha_i(x),1]$, and $a_i(j)=1$ if the i–th argument is $[0,\alpha_i(x))$. With such construction, the marginal distribution of U_i given $X=x$ is a uniform distribution on $[0,1]$ which satisfies Assumptions G.2 and G.3. Thus, the constructed structure $S\equiv[\pi_1,\cdots,\pi_I;F_{U|X}]\in\mathcal{M}_1'$ and it rationalizes $F_{Y|X}(\cdot|x)$ by construction. □

B.3. **Proof of Lemma 3.** Fix $x\in\mathscr{S}_X$. For any structure $S\in\mathcal{M}_1$, let $\tilde{F}_{U|X}(\cdot|x)=F_{U|X}(\cdot|x)$ and $\tilde{\pi}_i(a_{-i},x,u_i)=u_i^*(x)-u_i$ where $\left(u_1^*(x),\ldots,u_I^*(x)\right)$ is the equilibrium cut-off value profile under structure S. It is easy to see that $\tilde{S}\equiv[\tilde{\pi};\tilde{F}_{U|X}]\in\mathcal{M}_2$, and $\tilde{S}$ is observationally equivalent to the given structure S. □

B.4. **Proof of Lemma 4.** The only if part is straightforward: Note that $\alpha_i(x)=\mathbb{E}(Y_i|X=x)=\mathbb{P}(Y_i=1|X=x)$. Thus, given two observationally equivalent structures $S\equiv[\pi,F_{U|X}]$ and $\widetilde{S}\equiv[\tilde{\pi};\tilde{F}_{U|X}]$ in $\mathcal{M}_2$, condition (i) requires that both structures lead to the same value for $\mathbb{P}(Y_i=1|X=x)$, while condition (ii) requires to have the same value for $\mathbb{P}(Y_{i_1}=1,\cdots,Y_{i_p}=1|X=x)$.

For the if part, it suffices to verify that for every $x\in\mathscr{S}_X$, the conditional equilibrium choice probability $\mathbb{P}\left(Y=a|X=x\right)$ induced by the structure S can also be generated by another structure $\widetilde{S}$ that satisfies the two conditions in Lemma 4. We verify this for $a=(1,\cdots,1)$ only. By the definition of monotone pure strategy BNE, we have $\mathbb{P}_{\tilde{\delta}^*}\left(Y_1=\cdots=Y_I=1|X=x\right)=\tilde{F}_{U|X}\left(\tilde{u}_1^*(x),\cdots,\tilde{u}_I^*(x)\middle|x\right)$. By condition (i), $\tilde{u}_i^*(x)=\tilde{F}_{U_i|X}^{-1}\left(\alpha_i(x)\middle|x\right)$. Thus

$$\begin{aligned}
\mathbb{P}_{\tilde{\delta}^*}\left(Y_1=\cdots=Y_I=1|X=x\right) &= \tilde{F}_{U|X}(\tilde{F}_{U_1|X}^{-1}(\tilde{\alpha}_1(x)|x),\cdots,\tilde{F}_{U_I|X}^{-1}(\tilde{\alpha}_I(x)|x)|x)\\
&= \tilde{C}_{U|X}\left(\tilde{\alpha}_1(x),\cdots,\tilde{\alpha}_I(x)|x\right)\\
&= C_{U|X}\left(\alpha_1(x),\cdots,\alpha_I(x)|x\right)\\
&= \mathbb{P}_{\delta^*}\left(Y_1=\cdots=Y_I=1|X=x\right)
\end{aligned}$$

where the third equality follows from (ii). □

[13] To have meaningful partition, it is understood that $\{[0,\alpha_i(x)),[\alpha_i(x),1]\}$ becomes $\{\{0\},(0,1]\}$ when $\alpha_i(x)=0$.

B.5. **Proof of Lemma 5.** We first show the "only if" part. Suppose that the observationally equivalent structure is $\widetilde{S} = [\tilde{\pi}; \tilde{F}_{U|X}] \in \mathcal{M}_3$. Then $\tilde{F}_{U|X} = \tilde{F}_U$. Condition (ii) of Lemma 4 implies that

$$C_{U_{i_1},\cdots,U_{i_p}|X}\left(\alpha_{i_1}(x),\cdots,\alpha_{i_p}(x)\middle|x\right) = \widetilde{C}_{U_{i_1},\cdots,U_{i_p}}\left(\alpha_{i_1}(x),\cdots,\alpha_{i_p}(x)\right) \tag{11}$$

for every $x \in \mathscr{S}_X$, every $p = 2, \cdots, I$ and every i_js such that $1 \le i_1 < \cdots < i_p \le I$. Consequently, we have

$$\begin{aligned}
& m_p(\alpha_{i_1}(x),\cdots,\alpha_{i_p}(x)) \\
\equiv\ & \mathbb{E}\left[C_{U_{i_1},\cdots,U_{i_p}|X}\left(\alpha_{i_1}(X),\cdots\alpha_{i_p}(X)\middle|X\right)|\alpha_{i_1}(X) = \alpha_{i_1}(x),\cdots\alpha_{i_p}(X) = \alpha_{i_p}(x)\right] \\
=\ & \mathbb{E}\left[\widetilde{C}_{U_{i_1},\cdots,U_{i_p}}(\alpha_{i_1}(X),\cdots,\alpha_{i_p}(X))\middle|\alpha_{i_1}(X) = \alpha_{i_1}(x),\cdots,\alpha_{i_p}(X) = \alpha_{i_p}(x)\right] \\
=\ & \widetilde{C}_{U_{i_1},\cdots,U_{i_p}}(\alpha_{i_1}(x),\cdots,\alpha_{i_p}(x))
\end{aligned} \tag{12}$$

for every $x \in \mathscr{S}_X$. Condition (i) is then established by eqs. (11) and (12).

Notice that eq. (12) also implies that $m_p(\alpha_{i_1},\cdots,\alpha_{i_p}) = \widetilde{C}_{U_{i_1},\cdots,U_{i_p}}(\alpha_{i_1},\cdots,\alpha_{i_p})$ for every $(\alpha_{i_1},\cdots,\alpha_{i_p})$ in the support of $(\alpha_{i_1}(X),\cdots,\alpha_{i_p}(X))$. In addition, $\tilde{C}_{U_{i_1},\cdots,U_{i_p}}$ is monotone strictly increasing and continuously differentiable by Assumption R. Thus, condition (ii) holds due to the strict monotonicity of $\tilde{C}_{U_{i_1},\cdots,U_{i_p}}$, and condition (iii) can be obtained by the continuous differentiability of $\widetilde{C}_{U_{i_1},\cdots,U_{i_p}}$.

For the if part, our proof is constructive. For any $x \in \mathscr{S}_X$, let $\tilde{\pi}_i(a_{-i}, x, u_i) = \alpha_i(x) - u_i$ for $i = 1, \cdots, I$, where $\alpha_i(x) = F_{U_i|X}(u_i^*(x)|x)$. We further construct $\tilde{F}_U$. Let $\tilde{F}_{U_i}$ be uniformly distributed on $[0,1]$; And for all $1 \le i_1 < \cdots < i_p \le I$, $(\alpha_{i_1},\cdots,\alpha_{i_p}) \in \mathscr{S}_{\alpha_{i_1}(X),\cdots,\alpha_{i_p}(X)}$ and $x \in \mathscr{S}_X$, define $\widetilde{F}_{U_{i_1},\cdots,U_{i_p}}(\cdot,\cdots,\cdot)$ as follows

$$\widetilde{F}_{U_{i_1},\cdots,U_{i_p}}(\alpha_{i_1},\cdots,\alpha_{i_p}) = \mathbb{E}[C_{U_{i_1},\cdots,U_{i_p}|X}(\alpha_{i_1}(X),\cdots,\alpha_{i_p}(X)|X)\big|\alpha_{i_1}(X) = \alpha_{i_1},\cdots,\alpha_{i_p}(X) = \alpha_{i_p}],$$

which is monotone strictly increasing on $\mathscr{S}_{\alpha_{i_1}(X),\cdots,\alpha_{i_p}(X)}$ by condition (ii) and continuously differentiable in the interior of $\mathscr{S}_{\alpha_{i_1}(X),\cdots,\alpha_{i_p}(X)}$ by condition (iii). Thus we can extend the distribution $\tilde{F}_U(\cdot)$ to the whole support $[0,1]^I$ such that (1) $\tilde{F}_U(\cdot)$ is monotone strictly increasing and continuously differentiable on $[0,1]^I$; (2) $\tilde{F}_U(0,\cdots,0) = 0$ and $\tilde{F}_U(1,\cdots,1) = 1$. Thus $\tilde{F}_U(\cdot)$ is a proper distribution function and yields a positive and continuous conditional Radon–Nikodym density on $[0,1]^I$. By construction, $[\tilde{\pi}; \tilde{F}_U(\cdot)] \in \mathcal{M}_3$, and by Lemma 4 it is observationally equivalent to the structure S, because $\tilde{F}_{U_i}(\alpha_i(x)) = \alpha_i(x) = F_{U_i|X}(u_i^*(x)|x)$ where $(\alpha_1(x),\cdots,\alpha_I(x))$ is an equilibrium cutoff value profile under constructed structure $[\tilde{\pi}; \tilde{F}_U(\cdot)]$, and

$$C_{U_{i_1},\cdots,U_{i_p}|X}(\alpha_{i_1}(x),\cdots,\alpha_{i_p}(x)|x) = \tilde{F}_{U_{i_1},\cdots,U_{i_p}}(\alpha_{i_1}(x),\cdots,\alpha_{i_p}(x))$$

which is implied by condition (i). This completes the proof. □

B.6. **Proof of Proposition 3.**

Proof. Note that $C_{U_{i_1},\cdots,U_{i_p}|X}(\alpha_{i_1}(x),\cdots,\alpha_{i_p}(x)|x) = F_{U_{i_1},\cdots,U_{i_p}|X}\left(u^*_{i_1}(x),\cdots,u^*_{i_p}(x)|x\right) = \mathbb{E}\left(\prod_{j=1}^{p} Y_{i_j}\Big| X = x\right)$. In addition,

$$\begin{aligned}
& m_p(\alpha_{i_1}(x),\cdots,\alpha_{i_p}(x)) \\
= \;& \mathbb{E}\left[C_{U_{i_1},\cdots,U_{i_p}|X}\left(\alpha_{i_1}(X),\cdots\alpha_{i_p}(X)\big|X\right)|\alpha_{i_1}(X) = \alpha_{i_1}(x),\cdots\alpha_{i_p}(X) = \alpha_{i_p}(x)\right] \\
= \;& \mathbb{E}\left[\mathbb{E}\left(\prod_{j=1}^{p} Y_{i_j}|X\right)|\alpha_{i_1}(X) = \alpha_{i_1}(x),\cdots\alpha_{i_p}(X) = \alpha_{i_p}(x)\right] \\
= \;& \mathbb{E}\left[\prod_{j=1}^{p} Y_{i_j}|\alpha_{i_1}(X) = \alpha_{i_1}(x),\cdots\alpha_{i_p}(X) = \alpha_{i_p}(x)\right]
\end{aligned}$$

Thus, conditions (R1)-(R3) follow from (i)-(iii) in Lemma 5. □

B.7. **Proof of Lemma 6.**

Proof. The only if part follows directly from Assumption I. It suffices to show the if part.

We use a constructive approach to show the if part. Fix an arbitrary structure $[\pi; F_{U|X}] \in \mathcal{M}_3$ for which eq. (8) is satisfied. Fix arbitrarily $x \in \mathscr{S}_X$. Let $\tilde{F}_{U_i|X}(\cdot|x) = F_{U_i}(\cdot)$ and $\tilde{F}_{U|X}(\cdot|x) = \prod_{i=1}^{I} F_{U_i}(\cdot)$. Moreover, for any $x \in \mathscr{S}_X$, let $\tilde{\pi}_i(a_{-i}, x, u_i) = u^*_i(x) - u_i$ where $(u^*_1(x),\ldots,u^*_I(x))$ is the equilibrium cut-off value profile under the given structure $[\pi; F_{U|X}]$ and $X = x$. By construction, $[\tilde{\pi}; \tilde{F}_{U|X}]$ satisfies Assumptions M, S, E, and I. Regarding to Assumption R, it suffices to show that $F_{U_i}(\cdot)$ is absolutely continuous w.r.t. Lebesgue measure and has a continuous conditional Radon–Nikodym density $f_{U_i}(\cdot)$, which is true due to the fact $[\pi; F_{U|X}] \in \mathcal{M}_3$. Hence, $[\tilde{\pi}; \tilde{F}_{U|X}] \in \mathcal{M}_4$.

By construction, it is straightforward to show that the conditions (i) and (ii) in Lemma 4 are satisfied by the constructed structure $[\tilde{\pi}; \tilde{F}_{U|X}]$ and the given structure S, which ensures the observational equivalence between the two structures. □

B.8. **Proof of Proposition 4.**

Proof. Note that the condition given in Lemma 6 is stronger than those in Lemma 5, so the necessary and sufficient condition for a structure in $\mathcal{M}_2$ to be observationally equivalent to a structure in $\mathcal{M}_4$ is the same as the one in Lemma 6. In addition, we have $C_{U_{i_1},\cdots,U_{i_p}|X}\left(\alpha_{i_1}(x),\cdots,\alpha_{i_p}(x)|x\right) = F_{U_{i_1},\cdots,U_{i_p}|X}\left(u^*_{i_1}(x),\cdots,u^*_{i_p}(x)|x\right) = \mathbb{E}\left(\prod_{j=1}^{p} Y_{i_j}|X=x\right)$. Thus condition (8) becomes

$$E\left(\prod_{j=1}^{p} Y_{i_j}|X=x\right) = \prod_{j=1}^{p} E(Y_{i_j}|X=x)$$

for $p = 2,\cdots,I$, which implies that $Y_1,\cdots,Y_I$ are conditionally independent given X. □

B.9. **Proof of Lemma 7.**

Proof. Fix $X = x$. For notational brevity, we only show this lemma for $I = 3$. It is straightforward to generalize the following arguments to more general case with $I > 3$.

For $i = 1,2,3$, $j \neq i, k \neq i$ and $j < k$, let $\xi_{i,j}(x) \equiv \mathbb{P}(U_j \leq u^*_j(x)|X = x, U_i = u^*_i(x))$ and $\xi_{i,jk}(x) \equiv \mathbb{P}(U_j \leq u^*_j(x), U_k \leq u^*_k(x)|X = x, U_i = u^*_i(x))$. Note that for all $a_{-i} \in \mathcal{A}_{-i}$, $\sigma^*_{-i}(a_{-i}|x, u^*_i(x))$ can be expressed in terms of $\xi_{i,j}(x)$, $\xi_{i,k}(x)$ and $\xi_{i,jk}(x)$. For instance,

$$\sigma^*_{-1}(a_2 = 1, a_3 = 1|x, u^*_1(x)) = \mathbb{P}\big(U_2 \leq u^*_2(X); U_3 \leq u^*_3(X)|X = x, U_1 = u^*_1(x)\big) = \xi_{1,23}(x),$$

and

$$\sigma^*_{-1}(a_2 = 1, a_3 = 0|x, u^*_1(x)) = \xi_{1,2}(x) - \xi_{1,23}(x).$$

Thus, it suffices to show that all of $\xi_{i,j}(\cdot)$, $\xi_{i,k}(\cdot)$ and $\xi_{i,jk}(\cdot)$ are identified for all i, j, k such that $j, k \neq i$ and $j < k$.

Let $\alpha_i(x) \equiv F_{U_i|X}(u^*_i(x)|x)$ and $\alpha(x) \equiv (\alpha_1(x),\cdots,\alpha_I(x)) \in \mathbb{R}^I$. We then get $\alpha_i(x) = \mathbb{E}(Y_i|X = x)$ in equilibrium. Thus, $u^*_i(x) = F^{-1}_{U_i|X}\big(\alpha_i(x)|x\big) = F^{-1}_{U_i}\big(\alpha_i(x)\big)$, where the last step comes from Assumption E. Thus, in a monotone pure–strategy BNE,

$$\begin{aligned}\mathbb{E}(Y_iY_j|\alpha(X) = \alpha) &= \mathbb{P}\big(U_i \leq u^*_i(X); U_j \leq u^*_j(X)|\alpha(X) = \alpha\big)\\ &= \mathbb{P}\left(U_i \leq F^{-1}_{U_i}(\alpha_i); U_j \leq F^{-1}_{U_j}(\alpha_j)\Big|\alpha(X) = \alpha\right)\\ &= \mathbb{P}\left(U_i \leq F^{-1}_{U_i}(\alpha_i); U_j \leq F^{-1}_{U_j}(\alpha_j)\right).\end{aligned}$$

Hence, $\mathbb{E}(Y_iY_j|\alpha(X)=\alpha)$ is differentiable in $\alpha \in \mathscr{S}_{\alpha(X)}$ since $\mathbb{P}\left(U_i \le F_{U_i}^{-1}(\alpha_i); U_j \le F_{U_j}^{-1}(\alpha_j)\right)$ is differentiable in α under Assumption R.

$$\frac{\partial \mathbb{E}(Y_iY_j|\alpha(X)=\alpha)}{\partial \alpha_i}\Big|_{\alpha=\alpha(x)} = \frac{\partial \mathbb{P}\left(U_i \le F_{U_i}^{-1}(\alpha_i); U_j \le F_{U_j}^{-1}(\alpha_j)\right)}{\partial \alpha_i}\Big|_{\alpha=\alpha(x)}$$
$$= \frac{\partial \mathbb{P}\left(F_{U_i}(U_i) \le \alpha_i; F_{U_j}(U_j) \le \alpha_j\right)}{\partial \alpha_i}\Big|_{\alpha=\alpha(x)} = \frac{\partial \int_0^{\alpha_i}\int_0^{\alpha_j} f(\mu,\nu)d\nu d\mu}{\partial \alpha_i}\Big|_{\alpha=\alpha(x)},$$

where $f(\mu,\nu)$ is the joint density function of $\left(F_{U_i}(U_i), F_{U_j}(U_j)\right)$. Then

$$\frac{\partial \mathbb{E}(Y_iY_j|\alpha(X)=\alpha)}{\partial \alpha_i}\Big|_{\alpha=\alpha(x)} = \int_0^{\alpha_j(x)} f(\alpha_i(x),\nu)d\nu$$
$$= \int_0^1 f(\alpha_i(x),\nu)d\nu \times \int_0^{\alpha_j(x)} \frac{f(\alpha_i(x),\nu)}{\int_0^1 f(\alpha_i(x),\nu)d\nu}d\nu.$$

Note that $\int_0^1 f(\alpha_i(x),\nu)d\nu$ is the marginal density of $F_{U_i}(U_i)$ evaluated at $\alpha_i(x)$, which equals to one under Assumption R because $F_{U_i}(U_i)$ is uniformly distributed on $[0,1]$, and $f(\alpha_i(x),\nu)/\int_0^1 f(\alpha_i(x),\nu)d\nu$ is the conditional density function of $F_{U_j}(U_j)$ at ν given $F_{U_i}(U_i)=\alpha_i(x)$. Therefore,

$$\frac{\partial \mathbb{E}(Y_iY_j|\alpha(X)=\alpha)}{\partial \alpha_i}\Big|_{\alpha=\alpha(x)} = \int_0^{\alpha_j(x)} \frac{f(\alpha_i(x),\nu)}{\int_0^1 f(\alpha_i(x),\nu)d\nu}d\nu = \mathbb{P}\left(F_{U_j}(U_j) \le \alpha_j(x) \big| F_{U_i}(U_i) = \alpha_i(x)\right)$$
$$= \mathbb{P}\left(U_j \le F_{U_j}^{-1}(\alpha_j(x)) \big| U_i = F_{U_i}^{-1}(\alpha_i(x))\right) = \mathbb{P}\left(U_j \le u_j^*(x) \big| U_i = u_i^*(x)\right),$$

where the last step uses the fact that $u_i^*(x) = F_{U_i}^{-1}(\alpha_i(x))$. Hence, $\xi_{i,j}(\cdot)$ is identified. $\xi_{i,k}(\cdot)$ can be identified in the same way by replacing every subscript j with k. Similarly, we can obtain

$$\frac{\partial \mathbb{E}(Y_iY_jY_k|\alpha(X)=\alpha)}{\partial \alpha_i}\Big|_{\alpha=\alpha(x)} = \mathbb{P}\left(U_j \le u_j^*(x); U_k \le u_k^*(x) \big| U_i = u_i^*(x)\right),$$

which identifies $\xi_{i,jk}(\cdot)$. The desired conclusion therefore follows. □

B.10. **Model restrictions imposed by Assumption E.** Let $\mathcal{M}_5 \equiv \{S \in \mathcal{M}_3 : \text{ Assumptions RC–1 and ER hold}\}$.

Proposition 8. *For an arbitrary given structure $S \in \mathcal{M}_3$, suppose Assumption RC–1 holds. Then, there exists an observationally equivalent structure $\widetilde{S} \in \mathcal{M}_5$ if and only if $\forall w \in \mathscr{S}_W$, there exists a (conditional)* *probability distribution* $\mathbb{P}^o$ *and a profile of functions* $b(\cdot) \equiv (b_1(\cdot), \cdots, b_I(\cdot))$*, where* $b_i : \mathscr{S}_{Z_i|W=w} \to \mathbb{R}^{2^{I-1}}$ *and both* $\mathbb{P}^o$ *and* $b(\cdot)$ *could depend on* w*, such that*

(i) *Let* $Q_i^o(\cdot)$ *be the marginal quantile function of the i–th argument under the probability distribution* $\mathbb{P}^o$*. Then* $\forall x \in \mathscr{S}_{X|W=w}$*, there is*

$$\sum_{k=1}^{2^{I-1}} b_{ik}(z_i) \times \sigma_{-i}^*(a_{-i}^{k-1}|x, u_i^*(x)) = Q_i^o(\alpha_i(x)).$$

(ii) *The c.d.f. of* $\mathbb{P}^o$ *is continuously differentiable and monotone increasing in all arguments. Moreover,* $\forall p = 2, \cdots, I$*,* $\forall\, 1 \leq i_1 < \cdots < i_p \leq I$*, and* $\forall (\alpha_{i_1}, \cdots, \alpha_{i_p}) \in \mathscr{S}_{\alpha_{i_1}(X), \cdots, \alpha_{i_p}(X)|W=w}$*, there is*

$$\mathcal{C}^o_{i_1, \cdots, i_p}(\alpha_{i_1}, \cdots, \alpha_{i_p}) = \mathcal{C}_{U_{i_1}, \cdots, U_{i_p}|W}\left(\alpha_{i_1}, \cdots, \alpha_{i_p}|w\right),$$

where $\mathcal{C}^o_{i_1, \cdots, i_p}$ *be the copula function of the* i_1*–th,* $\cdots$*,* i_p*–th arguments under* $\mathbb{P}^o$*.*

(iii) $\forall \alpha \in [0,1]^I$*, let* $\delta_j(\cdot, \alpha_j) = \mathbf{1}(\cdot \leq Q_j^o(\alpha_j))$ *for all* $j = 1, \cdots, I$*. Let* $\delta_{-i} = (\delta_1, \cdots, \delta_{i-1}, \delta_{i+1}, \cdots, \delta_I)$*. Then for* $i = 1, \cdots, I$*, the function*

$$\sum_{k=1}^{2^{I-1}} b_{ik}(z_i) \times \mathbb{P}^o\left(\delta_{-i}(U_{-i}, \alpha_{-i}) = a_{-i}^{k-1}|U_i = u_i\right) - u_i$$

is a monotone decreasing function in u_i*.*

Proof. For the only if part. Suppose $[\pi; F_{U|W}] \in \mathcal{M}_3$ and $[\tilde{\pi}; \tilde{F}_{U|W}] \in \mathcal{M}_5$ are observationally equivalent. Fix $W = w$. Let $b_{ik}(z_i) = \tilde{\pi}_i(a_{-i}^k, w, z_i)$ and $\mathbb{P}^o$ be the conditional distribution of U given $W = w$ under $\tilde{F}_{U|W}(\cdot|w)$. Because of Lemma 2, $\forall x \in \mathscr{S}_{X|W=w}$

$$\sum_{a_{-i} \in \mathcal{A}} \tilde{\pi}_i(a_{-i}, w, z_i) \tilde{\sigma}_{-i}^*\left(a_{-i}|x, u_i^*(x)\right)) = \tilde{F}^{-1}_{U_i|W}(\tilde{\alpha}_i(x)|w).$$

By Lemma 7 and the observational equivalence between $[\pi; F_{U|W}]$ and $[\tilde{\pi}; \tilde{F}_{U|W}]$, $\tilde{\sigma}_{-i}^*\left(a_{-i}|x, u_i^*(x)\right)) = \sigma_{-i}^*\left(a_{-i}|x, u_i^*(x)\right))$ and $\tilde{\alpha}_i(x) = \alpha_i(x)$. Thus

$$\sum_{a_{-i} \in \mathcal{A}} b_{i,k}(z_i) \sigma_{-i}^*\left(a_{-i}|x, u_i^*(x)\right)) = \tilde{F}^{-1}_{U_i|W}(\alpha_i(x)|w).$$

By construction, $\tilde{F}^{-1}_{U_i|W}(\alpha_i(x)|w) = Q_i^o(\alpha_i(x))$. Thus condition (i) holds. Moreover, it is straightforward that condition (ii) and (iii) hold.

For the if part. We construct the structure $\tilde{S} \in \mathcal{M}_5$ as follows: under assumptions S and E, let $\tilde{\pi}_i(\cdot, w, z_i) = b_i(z_i)$ for all z_i, and $\tilde{F}_{U|X}(\cdot|x) = \tilde{F}_{U|W}(\cdot|w)$, which is derived from $\mathbb{P}^o$. By construction, it is straightforward that condition S, I and E are satisfied. Assumption R and MD are also satisfied by conditions (ii) and (iii). Thus, it is straightforward that $\tilde{\delta}^* = (\tilde{\delta}_1^*, \cdots, \tilde{\delta}_I^*)$, where $\tilde{\delta}_j^* = \mathbf{1}(u_j \leq Q_j^o(\alpha_j(x)))$, is a BNE in the constructed structure, which gives us observationally equivalent under the second part of condition (ii). □

Remark 2. *Condition (i) in Proposition 8 implies that*

a. *Condition on* $W = w$ *and* $Z_i = z_i$, $\sigma^*_{-i}\left(a^0_{-i}|X, u_i^*(X)\right), \cdots, \sigma^*_{-i}\left(a^{2^{I-1}-1}_{-i}|X, u_i^*(X)\right)$ *and* $Q_i^o(\alpha_i(X))$ *are linearly dependent, especially,* $\alpha_i(X)$ *is a linear–index function of* $\sigma^*_{-i}\left(a^0_{-i}|X, u_i^*(X)\right), \cdots, \sigma^*_{-i}\left(a^{2^{I-1}-1}_{-i}|X, u_i^*(X)\right)$.

b. *Condition on* $W = w$, $Z_i = z_i$ *and* $\alpha_i(X) = \alpha_i \in \mathscr{S}_{\alpha_i(X)|W=w}$, $\overline{\sigma}^*_{-i}\left(a^1_{-i}|X, u_i^*(X)\right), \cdots, \overline{\sigma}^*_{-i}\left(a^{2^{I-1}-1}_{-i}|X, u_i^*(X)\right)$ *are linearly dependent, where* $\overline{\sigma}^*_{-i}(a_{-i}|X, u_i^*(X)) = \sigma^*_{-i}(a_{-i}|X, u_i^*(X)) - \mathbb{E}\left[\sigma^*_{-i}(a_{-i}|X, u_i^*(X))|W = w, Z_i = z_i, \alpha_i(X) = \alpha_i\right]$.

B.11. **Proof of Proposition 6.**

Proof. When $\alpha_i(x) \in (0,1)$, because $\mathbb{E}_{\delta^*}\left[\pi_i(Y_{-i}, X_i, U_i)\big|X = x, U_i = u_i\right]$ is a continuous and monotone function in u_i by Assumptions R, M, and C, we then have

$$0 = \mathbb{E}_{\delta^*}\left[\pi_i(Y_{-i}, X_i, U_i)\big|X = x, U_i = u_i^*(x)\right] = \sum_{a_{-i}} \pi_i(a_{-i}, x_i, u_i^*(x)) \times \sigma_i^*(a_{-i}|x, u_i^*(x)).$$

Since $u_i^*(x) = q_i(\alpha_i(x))$, then we have

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \pi_i\big(a_{-i}, x_i, q_i(\alpha_i(x))\big) \times \sigma_i^*(a_{-i}|x, u_i^*(x)) = 0.$$

Let, w.l.o.g., the matrix $\mathbb{E}\left[\tilde{\Sigma}^*_{-i}(X)\tilde{\Sigma}^*_{-i}(X)^\top\big|X_i = x_i, \alpha_i(X) = \alpha_i\right]$ has a full rank that equals to $2^{I-1} - 1$, where $\tilde{\Sigma}^*_{-i}(x) \in \mathbb{R}^{2^{I-1}-1}$ defined by excluding $\sigma^*_{-i}\left(a^0_{-i}|x, u_i^*(x)\right)$ from $\Sigma^*_{-i}(x)$ with $a^0_{-i} \equiv (0, \cdots, 0)$. Then

$$\sum_{a_{-i} \in \mathcal{A}_{-i}/\{a^0_{-i}\}} \pi_i\big(a_{-i}, x_i, q_i(\alpha_i(x))\big) \times \sigma_i^*(a_{-i}|x, u_i^*(x)) = -\sigma_i^*(a^0_{-i}|x, u_i^*(x))\pi_i\big(a^0_{-i}, x_i, q_i(\alpha_i(x))\big).$$

By Lemma 7, $\sigma_i^*(a_{-i}|x, u_i^*(x))$ are identified for all a_{-i}. Thus, conditioning on x_i and $\alpha_i(x)$, the coefficient term $\pi_i(\cdot, x_i, q_i(\alpha_i(x)))$ in above equation is identified in terms of the unknown scale $\pi_i(a^0_{-i}, x_i, q_i(\alpha_i(x)))$ under Assumption RC–2.

Now we show the identification of the sign of $\pi_i(\cdot, x_i, q_i(\alpha_i))$. W.L.O.G., let $\alpha_i(x') < \alpha_i$. Then by Assumption M, there is

$$\sum_{a_{-i}\in\mathcal{A}_{-i}} \pi_i(a_{-i}, x_i, q_i(\alpha_i)) \times \sigma_i^*(a_{-i}|x, q_i(\alpha_i)) < 0.$$

By the proof of Lemma 7, $\sigma_i^*(a_{-i}|x, q_i(\alpha_i))$ is identified by the copula function of U when $(\alpha_i, \alpha_{-i}(x'))$ is in the support $\mathscr{S}_{\alpha(X)}$. Moreover, from above analysis, we know that $\pi_i(\cdot, x_i, q_i(\alpha_i))$ is identified up to the unknown scale $\pi_i(a^0_{-i}, x_i, q_i(\alpha_i(x)))$. Hence, we can denote

$$\pi_i(\cdot, x_i, q_i(\alpha_i)) = \overline{\pi}_i(\cdot, x_i, q_i(\alpha_i)) \times \pi_i(a^0_{-i}, x_i, q_i(\alpha_i(x)))$$

where $\overline{\pi}_i$ is the identified part. Thus

$$\left\{\sum_{a_{-i}\in\mathcal{A}_{-i}} \overline{\pi}_i(a_{-i}, x_i, q_i(\alpha_i)) \times \sigma_i^*(a_{-i}|x, q_i(\alpha_i))\right\} \times \pi_i(a^0_{-i}, x_i, q_i(\alpha_i(x))) < 0,$$

from which we identify the sign of $\pi_i(a^0_{-i}, x_i, q_i(\alpha_i(x)))$. □

B.12. **Proof of Proposition 7.** With additive separability in the payoffs as well as Assumption ER, we can obtain the following equilibrium condition by Lemma 2,

$$\pi_i\left(a^0_{-i}, x_i\right) + \sum_{a_{-i}\in\mathcal{A}_{-i}/\{a^0_{-i}\}} h_i(a_{-i}, x_i)\sigma^*_{-i}(a_{-i}|x, u_i^*(x))) = u_i^*(x) = q_i(\alpha_i(x)) \tag{13}$$

for all $x \in \mathscr{S}_X$ such that $\alpha_i(x) \in (0,1)$, where $h_i(a_{-i}, x_i) \equiv \pi_i(a_{-i}, x_i) - \pi_i(a^0_{-i}, x_i)$. Such an equilibrium condition implies that

$$\pi_i\left(a^0_{-i}, x_i\right) + \sum_{a_{-i}\in\mathcal{A}_{-i}/\{a^0_{-i}\}} h_i(a_{-i}, x_i)\mathbb{E}\left[\sigma^*_{-i}(a_{-i}|X, u_i^*(X)))|X_i = x_i, \alpha_i(X) = \alpha_i(x)\right] = q_i(\alpha_i(x)) \tag{14}$$

The difference between eq. (13) and (14) yields

$$\sum_{a_{-i}\neq a^0_{-i}} h_i(a_{-i}, x_i)\overline{\sigma}^*_{-i}(a_{-i}, x) = 0 \tag{15}$$

where $\overline{\sigma}^*_{-i}(a_{-i}, x) \equiv \sigma^*_{-i}(a_{-i}|x, u_i^*(x)) - \mathbb{E}\left[\sigma^*_{-i}(a_{-i}|X, u_i^*(X))|X_i = x_i, \alpha_i(X) = \alpha_i(x)\right]$.

Under rank condition RC–3, $h_i(\cdot, x_i)$ in eq. (15) is then identified up to scale. Moreover, the identification of the sign of $h_i(\cdot, x_i)$ can be shown similarly as that in the proof of Proposition 6. □

B.13. **Proof of Corollary 2.**

Proof. Fix $x_i \in \mathscr{S}_{X_i}$. Note that if $\pi_i(a_{-i}, x_i) = 0$ for all $a_{-i} \in \mathcal{A}_i$, then $\mathscr{S}_{\alpha_i(X)|X_i=x_i}$ will be a singleton. Then, w.l.o.g., let $\pi_i(a^0_{-i}, x_i) \neq 0$. Let, w.l.o.g., $h_i(a_{-i}, x_i) = \overline{h}_i(a_{-i}, x_i) \times h_i(a^1_{-i}, x_i)$ for $a_{-i} \in \mathcal{A}_{-i}/\{a^0_{-i}\}$, where $\overline{h}_i(a_{-i}, x_i)$ is the identified part. Then for some $x \in \mathscr{S}_{X|X_i=x_i}$, $\alpha_i(x) = 0$. Therefore,

$$\pi_i(a^0_{-i}, x_i) + \left\{\sum_{a_{-i}\in\mathcal{A}_{-i}/\{a^0_{-i}\}} \overline{h}_i(a_{-i}, x_i)\sigma^*_{-i}(a_{-i}|x, u_i^*(x))\right\} \times h_i(a^1_{-i}, x_i) = 0,$$

which implies that

$$\pi_i(a_{-i}, x_i) = -\frac{\overline{h}_i(a_{-i}, x_i) - \left\{\sum_{a_{-i}\in\mathcal{A}_{-i}/\{a^0_{-i}\}} \overline{h}_i(a_{-i}, x_i)\sigma^*_{-i}(a_{-i}|x, u_i^*(x))\right\}}{\left\{\sum_{a_{-i}\in\mathcal{A}_{-i}/\{a^0_{-i}\}} \overline{h}_i(a_{-i}, x_i)\sigma^*_{-i}(a_{-i}|x, u_i^*(x))\right\}} \times \pi_i(a^0_{-i}, x_i),$$

is identified up to the scale term $\pi_i(a^0_{-i}, x_i)$. Therefore, $\pi_i(a^0_{-i}, x_i)$ can be identified by the scale normalization $\|\pi_i(\cdot, x_i)\| = 1$, which further gives us $\pi_i(a_{-i}, x_i)$ for all $a_{-i} \in \mathcal{A}_{-i}$. □

B.14. **Necessary and sufficient condition for two structures in $\mathcal{M}_3$ to be observationally equivalent.**

Lemma 10. *Suppose that two structures $S \equiv [\pi; F_U]$ and $\widetilde{S} \equiv [\tilde{\pi}; \tilde{F}_U]$ in $\mathcal{M}_3$ satisfy Assumptions RC–1, ER and RC–3 for all $x \in \mathscr{S}_X$, then they are observationally equivalent if and only if (a) $C_U = \tilde{C}_U$ on $\mathscr{S}_{\alpha(X)}$; (b) there exist $\beta \in \mathbb{R}^I$ and $\gamma \in \mathbb{R}^I_+$, such that for all $i, a_{-i} \in \mathcal{A}_{-i}$ and $x_i \in \mathscr{S}_{X_i}$, $\tilde{\pi}_i(a_{-i}, x_i) = \beta_i + \gamma_i \times \pi_i(a_{-i}, x_i)$ and $\tilde{F}_{U_i}(\cdot) = F_{U_i}(\frac{\cdot-\beta_i}{\gamma_i})$ on the support $\mathscr{S}_{\beta_i+\gamma_i\times u_i^*(X)}$.*

Proof. The if part is straightforward, its proof is therefore omitted.

Now we show the only if part. Condition (a) follows immediately from condition (ii) of Lemma 4, so it suffices to show condition (b). First, we construct an observationally equivalent structure $\left[\pi^e_1(\cdot,\cdot), \cdots, \pi^e_I(\cdot,\cdot); F^e_U(\cdot)\right]$ of structure S, such that $F^{e-1}_U(\tau_{i1}) = 0$ and $F^{e-1}_U(\tau_{i2}) = 1$ for $\tau_{i1} < \tau_{i2} \in \mathscr{S}_{\alpha_i(X)|X_i=x_i}$ and some $x_i \in \mathscr{S}_{X_i}$. Let $C^e_{U_{i_1},\cdots,U_{i_p}} = C_{U_{i_1},\cdots,U_{i_p}}$ for all $p \geq 2$, $F^e_{U_i}(t) = F_{U_i}\left([t - \beta^e_i]/\gamma^e_i\right)$

and $\pi_i^e = \beta_i^e + \gamma_i^e \pi_i$, where

$$\beta_i^e = \frac{-F_{U_i}^{-1}(\tau_{i1})}{F_{U_i}^{-1}(\tau_{i2}) - F_{U_i}^{-1}(\tau_{i1})}, \quad \gamma_i^e = \frac{1}{F_{U_i}^{-1}(\tau_{i2}) - F_{U_i}^{-1}(\tau_{i1})} > 0.$$

Then, by Lemma 4, it is straightforward that $\left[\pi_1^e(\cdot,\cdot), \cdots, \pi_I^e(\cdot,\cdot); F_U^e(\cdot)\right]$ and structure S are observationally equivalent to each other . Moreover, $F_{U_i}^e(0) = \tau_{i1}$ and $F_{U_i}^e(1) = \tau_{i2}$.

Similarly, let $C_{U_{i_1},\cdots,U_{i_p}}^{\tilde{e}} = \tilde{C}_{U_{i_1},\cdots,U_{i_p}}$ for all $p \geq 2$, $F_{U_i}^{\tilde{e}}(t) = \tilde{F}_{U_i}\left([t - \beta_i^{\tilde{e}}]/\gamma_i^{\tilde{e}}\right)$ and $\pi_i^{\tilde{e}} = \beta_i^{\tilde{e}} + \gamma_i^{\tilde{e}} \tilde{\pi}_i$, where

$$\beta_i^{\tilde{e}} = \frac{-\tilde{F}_{U_i}^{-1}(\tau_{i1})}{\tilde{F}_{U_i}^{-1}(\tau_{i2}) - \tilde{F}_{U_i}^{-1}(\tau_{i1})}, \quad \gamma_i^{\tilde{e}} = \frac{1}{\tilde{F}_{U_i}^{-1}(\tau_{i2}) - \tilde{F}_{U_i}^{-1}(\tau_{i1})} > 0.$$

Then $\left[\pi_1^{\tilde{e}}(\cdot,\cdot), \cdots, \pi_I^{\tilde{e}}(\cdot,\cdot); F_U^{\tilde{e}}(\cdot)\right]$ is an observationally equivalent structure of structure $\tilde{S}$ such that $F^{\tilde{e}}{}_U^{-1}(\tau_{i1}) = 0$ and $F^{\tilde{e}}{}_U^{-1}(\tau_{i2}) = 1$. Thus, $\left[\pi_1^{\tilde{e}}(\cdot,\cdot), \cdots, \pi_I^{\tilde{e}}(\cdot,\cdot); F_U^{\tilde{e}}(\cdot)\right]$ is also observationally equivalent to structure S because of observational equivalence between structures S and $\tilde{S}$.

There is, however, only one observationally equivalent structure S^e of structure S satisfying $F_{U_i}^{e-1}(\tau_{i1}) = 0$ and $F_{U_i}^{e-1}(\tau_{i2}) = 1$. The reason is given as follows: By Proposition 7, $\pi_i^e(\cdot, x_i)$ is identified up to location and scale; And for some $x_{-i}, x'_{-i} \in \mathscr{S}_{X_{-i}|X_i=x_i}$, there is $\alpha_i(x) = \tau_{i1}$ and $\alpha_i(x') = \tau_{i2}$. Then

$$\sum_{a_{-i}} \pi_i^e(a_{-i}, x_i) \times \sigma_{-i}^*(a_{-i}|X = x, U_i = u_i^*(x)) = F_{U_i}^{e-1}(\tau_{i1}) = 0,$$

$$\sum_{a_{-i}} \pi_i^e(a_{-i}, x_i) \times \sigma_{-i}^*(a_{-i}|X = x', U_i = u_i^*(x')) = F_{U_i}^{e-1}(\tau_{i2}) = 1,$$

from which we obtain a unique location and scale for $\pi_i^e(\cdot, x_i)$.

Consequently, $\pi_i^e(a_{-i}, x_i) = \beta_i^e + \gamma_i^e \pi_i(a_{-i}, x_i) = \beta_i^{\tilde{e}} + \gamma_i^{\tilde{e}} \tilde{\pi}_i(a_{-i}, x_i) = \pi_i^{\tilde{e}}(a_{-i}, x_i)$ for all a_{-i} and x_i, from which we have

$$\tilde{\pi}_i(\cdot, x_i) = \beta_i + \gamma_i \pi_i(\cdot, x_i).$$

where $\beta_i = \frac{\beta_i^e - \beta_i^{\tilde{e}}}{\gamma_i^{\tilde{e}}}, \gamma_i = \frac{\gamma_i^e}{\gamma_i^{\tilde{e}}}$. Besides, in equilibrium, we can obtain the following conditions for structure S and $\tilde{S}$, respectively.

$$\sum_{a_{-i}} \pi_i(a_{-i}, x_i) \cdot \sigma_{-i}^*(a_{-i}|x, u_i^*(x)) = u_i^*(x)$$

$$\sum_{a_{-i}} \tilde{\pi}_i(a_{-i}, x_i) \cdot \tilde{\sigma}_{-i}^*(a_{-i}|x, \tilde{u}_i^*(x)) = \tilde{u}_i^*(x)$$

which imply $\tilde{u}_i^*(x) = \beta_i + \gamma_i \cdot u_i^*(x)$ since we have $\tilde{\pi}_i(\cdot, x_i) = \beta_i + \gamma_i \pi_i(\cdot, x_i)$ obtained earlier and $\sigma_{-i}^*(\cdot|x, u_i^*(x)) = \tilde{\sigma}_{-i}^*(\cdot|x, \tilde{u}_i^*(x))$ from condition (a) and Lemma 7. We then get $\tilde{F}_{U_i}(u_i) = F_{U_i}(\frac{u_i - \beta_i}{\gamma_i})$ for every $u_i \in \mathscr{S}_{\beta_i + \gamma_i \cdot u_i^*(X)}$ from the observational equivalence condition $F_{U_i}(u_i^*(x)) = \tilde{F}_{U_i}(\tilde{u}_i^*(x))$, which completes the proof. $\square$